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Fundamental dynamics and control strategies for aseismic structural control

Kazuhiko Yamada *, Takuji Kobori [1]

Kobori Research Complex, Kajima Corporation, KI Building, 6-5-30, Akasaka, Minato-ku, Tokyo 107-8502, Japan

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Abstract

This paper introduces fundamental dynamics and control strategies for aseismic structural control, especially focusing on excitation influence and nonlinear effects. Firstly, fundamental structural dynamics with control forces under a seismic excitation are introduced. Then, as a basic strategy for structural control, the effects of linear feedback control laws are reviewed. Next, introducing the least quadratic regulator (LQR) considering excitation influence, the least input energy control and the LQR for short term under given excitation information, roles of the FB, instantaneous counter-reaction and feedforward (FF) terms are clarified. Furthermore, it is inferred that it is difficult to explicitly solve the optimization problems positively considering control force limit, and the Euler equations for the optimal variable-element control become nonlinear. It is also shown that we can construct an extended FB control law assuming a state equation model for excitation information. Then, not only an additional damping effect but also a dissonant effect on a seismic excitation are anticipated. Furthermore, sufficient stability conditions for nonlinear control laws are introduced. Finally, as examples of nonlinear control laws, nonlinear velocity FB laws and nonlinear Maxwell-type control laws are introduced.

Keywords: Structural control; Seismic excitation; Feedback; Feedforward; Nonlinear control; Optimization; Stability

1. Introduction

1.1. Background

A large catfish deep in the earth quakes the ground, thus shaking houses. Even when people believed it, carpenters knew that a tall pagoda is imparted less motion from the ground than a small house, and that complicated combinations of wooden elements resolve its sway. Even for an ambiguous earthquake, excitation influence and nonlinear effects were considered in structural design a thousand years and more ago.

* Corresponding author. Tel.: +81-3-5561-2423; fax: +81-3-5561-2431.
E-mail address: kazu@krc.kajima.co.jp (K. Yamada).
[1] Professor Emeritus, Kyoto University.

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Later, Newtonian mechanics served engineers a mighty card. Based on the relativity principle, seismic influence was expressed as an inertia force caused by ground movement. Because the size of ground movement was still indefinite, the inertia force scale was guessed as a certain ratio to the gravity. Aseismic elements such as stiff shear walls were installed in structures to statically resist lateral loads due to the assumed inertia force. These aseismic elements were designed to be linear for lateral loads. A simple calculation method rather than purely experience was applied to complex structures, which enabled structures to be comprised of frames of several spans. However, the structure should be stiff so as not to largely deform under lateral loads. Thus, a high or long-span structure could not be designed.

Engineers studied dynamical excitation influence based on the linear vibration theory. A seismic excitation possessing large power at a structure's natural frequency produces enlarged structural responses, due to resonance to the seismic excitation. However, if a structure's natural frequency is dissonant to the excitation's dominant frequency, it is less influenced by the seismic excitation. Structural responses largely depend on the dynamical characteristics of the seismic excitation. Spectrum analyses of the seismic record for El Centro, California for the Imperial Valley earthquake in 1940 showed that a flexible tall building is exposed to less excitation influence because of its long natural period. This meant that we could build a high-rise building even in Japan.

The first high-rise building in Japan, the Kasumigaseki Building, has nonlinear elements, i.e., slit walls, which behave inelastically and follow the large deformations of a flexible building and resists lateral loads due to a seismic excitation. This idea is reflected in recent design codes, which allow buildings to experience inelastic deformation under extremely large earthquakes. Progress in computation has also helped to take into account inelasticity in design. By regarding a seismic load as a dynamic excitation, structural responses to an assumed seismic excitation can be simulated from time to time today.

Many special devices as well as structural elements are nowadays installed in many structures, aimed at reducing seismic response. Passive control devices such as elastic–plastic metal devices are used for energy dissipation by nonlinear hysteretic damping. Devices such as base isolation systems are anticipated to reduce responses by a dissonant effect to seismic excitation. Active control systems using mechanical devices such as actuators can also reduce structural responses. However, if they follow a linear control law, they require large external power and a large control system for structural control under large earthquakes. In practice, such large external power and control system cannot be provided. Thus, an active control system should be developed to consider various constraints. Otherwise, semi-active control systems, which produce controlled forces by changing parameters, will substitute for them. If we can deliberately change structural stiffness, a structure will be exposed to less energy from a ground quake. If the damping coefficient of a damping element can be changed, a load in a structural element could be adjusted to satisfy mechanical constraints. Structural dynamics and a control law for them are then nonlinear.

Regardless of whether the system is passive, active or semi-active, aseismic structural control enables design freedom. Because a control system can manage most lateral loads caused by a seismic excitation, structural elements should support only vertical loads due to gravity. Thus, beams and columns could be slender. A long-span space or a super-high-rise building can be planned. Simple design details for each structural element would improve construction productivity. Structural control serves us with not only mechanical advantages but also benefits in design and construction. Furthermore, structural control is indispensable for the future urban society. The Hyogoken–Nanbu earthquake in 1992 demonstrated to us that the function of an urban society should be immediately restored even after an extremely large earthquake. Otherwise, the earthquake will influence not only the immediately damaged area but also our global society. We should not allow irremediable structural damage. The future urban society should maintain its functions under any earthquake, using structural control.

However, research and development (R&D) on structural control to a seismic excitation is not a simple project because of excitations' ambiguity and scale. In most R&D on structural control, linear control theories for stationary or free vibration states have been applied. Thus, this has inevitably created a contradiction. To overcome this problem, we should consider excitation influence and nonlinear effects as the ancient carpenters did.

1.2. Subjects for aseismic structural control

Based on the above, the following subjects need to be considered in developing aseismic structural control strategies:

(1) A seismic excitation must be regarded as a nonstationary, narrow-banded wave with uncertainty. It is never just a disturbance or a white noise. Neither is it a stationary wave. Its dynamical characteristics greatly influence structural dynamics.

(2) Lateral loads of up to 1 g influence a structure in an extremely large seismic event. The total weight of a building is generally thousands of tons. According to the present design code, a building must be in the elastic range under lateral loads due to a seismic excitation, which amounts to about 20% of its weight at the basement. The response at the top of the building is in most cases amplified to two to five times that at the bottom. We must develop control systems that can control these loads.

(3) A seismic excitation of a few seconds to a few minutes is relatively short, compared to structural natural periods, so that a control force must quickly act on the structure. As the primary purpose, we should reduce maximum structural responses to a seismic excitation, aiming at avoiding structural collapses and reducing lateral design loads. However, the maximum responses occur near maximum excitations and there is not enough time to produce stationary resonant states.

(4) A structure such as a high-rise building is flexible. Many degrees of freedom (DOFs) are required to express structural motion under a seismic excitation. For example, to analyze the motion of a 50-story building in three dimensions, at least 150 DOFs are necessary, resulting in heavy computational tasks.

(5) Structural stiffness, weight, and damping involve uncertainty. These values for a completed structure, confirmed by a vibration test, differ from the design values. The partition wall stiffness may shorten structural natural periods of small amplitude, while yield of structural elements would lengthen those of large amplitude. Furthermore, weights are variable at each floor. Thus the control strategies are required to be robust.

Therefore, we must develop aseismic structural control strategies, (a) reflecting dynamic characteristics of excitation, (b) using a control system considering structural scale and requiring a small amount of external energy, (c) quickly reducing maximum responses in a transient state, (d) requiring few computational tasks, (e) being robust against fluctuations of structural characteristics.

1.3. Previous works

In 1950s, the authors (Kobori and Minai, 1955a,b; Kobori, 1956; Kobori and Minai, 1960a,b) introduced basic ideas of structural control, based on the nonlinear vibration theory. That is, to reduce structural responses to a seismic excitation, we: (1) shut-off energy transmission routes subjected to a seismic excitation; (2) isolate frequency bands of structural nature from those of a seismic excitation; (3) utilize a nonstationary and dissonant system with nonlinear characteristics; (4) use an energy dissipation mechanism; and (5) produce a control force on the structure. It was also inferred that inevitable nonlinear behaviors, as represented by elements' yielding, can avoid a structural collapse due to an earthquake, while artificial nonlinear procedures can ensure even structural functions against it. Furthermore, based on these ideas, combining a pretension wire and a twisted wire, which possess softening and hardening characteristics, respectively, can reduce structural responses, although mechanical technology in that period could not realize it. However, fundamental attitudes to control strategy development do not differ from these ideas even today.

From the end of the 1960s to the 1970s, more concrete ideas were proposed along with advanced mechanical and analytical technologies. Mahmoodi (1969) and Kelly et al. (1972) showed that we can install special devices as well as structural elements to reduce structural responses to seismic excitations. Yao (1972) introduced the idea of reducing structural responses by an external force, which controls a civil engineering structure like a machine. Yang (1975) applied the modern control theory to a civil engineering structure. Rooda (1975) assumed tendon control systems in tall structures. Martin and Soong (1976) examined modal control of structures. Abdel-Rohman and Leipholz (1978) proposed control strategies using a pole assignment method. Masri et al. (1981) studied pulse control of structures.

Later, as reviewed by Soong (1988), R&D of structural control has proceeded vigorously, especially since the 1980s. Soong (1990), Kobori (1993a), and Soong and Darguch (1997) summarized theories and practical applications at that time. Kobori et al. (1986) called a response controlled structure a dynamic intelligent building, comparing it to a human body. As the most fruitful achievement, the first actual actively-response-controlled building, using active mass dampers, was completed in 1989. Kobori et al. (1991a,b) confirmed that the active mass dampers reduce lateral and torsional swaying under small seismic and wind excitations. Successively, in 1990, Kobori et al. (1993b) installed an active variable stiffness system in an actual building to reduce structural responses to seismic excitations, anticipating a dissonant effect.

R&D in the 1990s has focused on making control systems more effective under large seismic excitation and more applicable to any structure. Housner et al. (1994) and Kobori (1996) showed the directions of future R&D on structural control. Kobori et al. (1992) also claimed that structural control technology is indispensable for a super-high-rise building. Housner et al. (1997) reviewed semi-active systems, active–passive hybrid systems, nonlinear systems and new material dampers. We should note that the structural dynamics with these developing systems have become so complex that advanced control strategies are required. Along with these streams, this paper reviews fundamental control strategies and then introduces the directions for developing control strategies, especially considering excitation influence and nonlinear effects.

In practice, nonlinear control strategies are adopted when various constraints must be considered. For example, the control force must be influenced by the control system. It may follow variable gain laws, or possess saturated values. Kobori et al. (1991b) proposed a simulation method for structural responses to seismic excitations, considering the control system's dynamics and the nonlinear element's influence. Tamura et al. (1994), Nagashima and Shinozaki (1997) and Yamamoto and Suzuki (1998) applied variable gain control to mass dampers. Nakagawa and Asano (1995) and Niwa et al. (1995) examined cases where maximum control forces are limited. Nishitani and Nitta (1998) and Bharta et al. (1994) considered mass dampers' constraints. Wu and Soong (1996), Mongkol et al. (1996), and Chase et al. (1996) examined the structural performance where control forces acting between stories have saturated values. Yang et al. (1996) and Tomasula et al. (1996) introduced optimal control laws, assuming higher-order norms. Yamada (1998) also studied the performance of more general nonlinear velocity feedback (FB) laws. Another approach is to follow hysteretic rules. Yang et al. (1994a) proposed a control force that follows a hysteretic rule. Yang et al. (1995) also introduced a hysteretic control force based on a sliding mode control. Soong (1998) described an algorithm for experimental simulation of a control force that plots Bouc–Wen hysteretic curves (Bouc, 1971; Wen, 1976). Bani-Hani and Ghaboussi (1998) introduced a hysteretic control force based on neural network theory. Yamada (2000) proposed a simpler control law based on a differential model for hysteresis.

Semi-active control systems are more practical because they require little external power. Chung et al. (1989), Ikeda and Kobori (1991), Tachibana et al. (1994), Loh and Ma (1994) and Yamada and Kobori (1995) theoretically and experimentally demonstrated that a variable stiffness structure can reduce structural responses to seismic excitations. Furthermore, Kobori et al. (1993b) demonstrated the control effect of the variable stiffness elements (VSEs) installed in an actual building. A variable damping element (VDE) can also control a reactive internal force in the range of the VDEs capacity. Feng and Shinozuka (1990) first discussed VDEs' effect on structural responses. Then, Kawashima et al. (1992), Kurata et al. (1994), Patten et al. (1994) and Symans et al. (1994) developed actual VDEs and examined their performances experimentally. Kurino et al. (1996) and Haroun et al. (1994) modeled VDEs' dynamics and examined their effects by simulation analyses. Polak et al. (1994) evaluated achievable and acceptable effects by VDEs. Sadek and Mohraz (1998) showed that VDEs' effects depend on structural natural frequency. To be more effective, Hayen and Iwan (1994), Iwan and Wang (1996) proposed nonlinear control laws for the VDEs based on more energy dissipation. Hatada and Smith (1997) introduced a control law for VDEs, considering their maximum capacity. Yamada (1999a) also proposed a nonlinear control law by adding a simple nonlinear term into the control force vs. velocity relation. We should note that dynamics considering these semi-active control systems are nonlinear.

Therefore, the more practical the control system, the more its nonlinearity should be considered. That is, to develop a more practical control system, a deeper knowledge of nonlinear dynamics and control strategies is required. The recent mathematical methods and results on nonlinear dynamics as shown in many literatures such as those by Guckenheimer and Holmes (1983), Wiggins (1990) and Jackson (1991) tell us that even a small nonlinear term may globally produce quite complex phenomena. In other words, it may be possible to express complex phenomena by a simple nonlinearity. Moon (1992) showed many examples of complex phenomena in mechanics caused by nonlinearity. We can not only mathematically analyze such complex phenomena, but also positively apply them to engineering. As is well known, the Van del Pole oscillator produces a stationary vibration as a disturbance subjected to nonlinearity. In aseismic structural control, we expect nonlinearity to be effectively taken into account in structural dynamics, thus realizing our design purposes under practical constraints. Regardless of the object for which we adopt nonlinearity, the variational principle (Arnonld, 1978), the maximum principle by Pontryagin et al. (1961) and the dynamic programming by Bellman (1957) are fundamental for developing a control strategy. The Lyapunov function, as introduced by Lasalle and Lefshetz (1961), is useful for examining stability and constructing a stable control law. Popov's criterion and the circle criterion (Kahlil, 1992), are fundamental even for structural control. To develop a nonlinear control strategy, the indirect control method by Lefshetz (1965), which expresses a control strategy by a differential equation, is also useful. Therefore, this paper introduces fundamental control strategies, applying these procedures to aseismic structural control.

Simultaneously, to make more thorough use of control system ability, research that considers more excitation influence as well as nonlinear effects has proceeded. The least quadratic regulator (LQR) is widely used and many active control systems basically follow it. This is because it is not only theoretically clear and robust but also explicitly provides states at a certain moment, which means that structural responses to a seismic excitation can be obtained. However, the LQR is induced assuming that the excitation term is neglected or of a white noise. For aseismic structural control, a better strategy should be developed. Thus, to reduce excitation influence, Smith and Chase (1994), Yoshida et al. (1994), Yang et al. (1994b), Jabbari et al. (1996), Yamada and Nishitani (1996), and Köse et al. (1996) applied H_∞ control law theory. While, we can more positively consider excitation information. Yang et al. (1987) proposed instantaneous optimal control, considering instantaneous excitation influence. Nishimura et al. (1992) and Sato et al. (1994), Wu et al. (1994), and Dyke et al. (1994) proposed acceleration as well as velocity FB. Suhardjo and Spencer (1990) proposed a feedback–feedforward control algorithm (feedback–feedforward, FB–FF), introducing a filter for the excitation. Naraoka and Katsukura (1992) studied the response characteristics when such FB–FF control algorithms controlled a structure. Shing et al. (1996) compares the FB–FF control with FB control. To take full account of future excitation influence, Fukazawa and Kawahara (1988) introduced a control law assuming that all future excitation information has been provided. However, this is not practical. Then, using the online identified AR model of excitation information, Yamada and Kabori (1996) extended the LQR. Thus, future excitation influence is taken account of in a practical way. Yamada and Kobori (1994) also showed that a control law minimizing input energy requires future excitation information. This paper reviews these strategies and examines them considering excitation influence.

1.4. Outlines

In response to these subjects, Section 2 first presents fundamental dynamics for aseismic structural control. It then introduces momentum equations and their modally decomposed forms, the state equation and its discrete form, and the energy balance equation, and discusses how the control force influences structural dynamics. Section 3 reviews the formulation and control effects of linear FB control laws, which are fundamental for any control strategies. To obtain hints on how to determine the gains for the linear FB laws, Section 4 examines optimal control laws for a seismic excitation, assuming that future excitation information is already known. However, this is shown to be impractical, so Section 5 introduces the extended FB control laws and the extended LQR (ELQR), assuming a state equation model of excitation information. Next, Section 6 examines optimization problems for cases positively considering control force limit and variable elements. Furthermore, Section 7 introduces sufficient stability conditions for nonlinear control laws. Lastly, as examples of nonlinear control laws, Section 8 introduces nonlinear velocity FB laws and nonlinear Maxwell (NMW)-type control forces.

2. Structural dynamics under seismic excitation

2.1. Control systems for aseismic structural control

Fig. 1 shows various control systems for reducing seismic structural responses. Active mass damper systems, active base isolation systems and active tendon systems provide control forces, using hydraulic actuators or servo motors. With an active mass damper system, a quickly moving small mass imparts a control force to a structure. With the active base isolation system, the ground supports the reaction to the control force. With the active tendon system, one control force is imparted at two points in a structure by the action and reaction principle. These control systems require not a little external power. However, semi-active control by variable elements such as VSEs, switching stiffness elements, VDE with auxiliary stiffness elements (ASEs), and so on, can impart large controlled forces with little external power. Because these

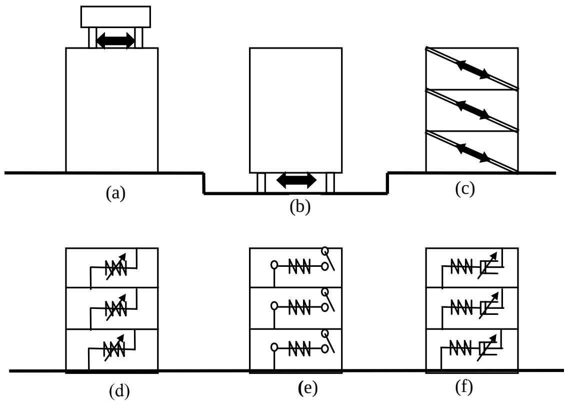

Fig. 1. Various control systems: (a) active mass damper system; (b) active base isolation system; (c) active tendon system; (d) VSEs; (e) switching stiffness elements; (f) VDE with ASE.

elements provide structural natural frequency change as well as hysteretic damping, we can expect them not only to dissipate structural vibration energy but also to expose a structure to little seismic energy.

2.2. Structural dynamics

Let us consider an nDOF structural model with m control forces. The structural dynamics is then expressed as:

$$\boldsymbol{M}\boldsymbol{y}''(t) + \boldsymbol{C}\boldsymbol{y}'(t) + \boldsymbol{K}\boldsymbol{y}(t) = -\boldsymbol{M}\boldsymbol{V}w(t) + \boldsymbol{U}\boldsymbol{u}(t), \tag{2.1}$$

where $\boldsymbol{y}''(t)$, $\boldsymbol{y}'(t)$ and $\boldsymbol{y}(t) \in \mathbb{R}^n$ represent the structural acceleration, velocity and displacement at time t relative to the structural basement, respectively; $w(t) \in \mathbb{R}$ represents the acceleration at the structural basement excited by a seismic event; $\boldsymbol{u}(t) \in \mathbb{R}^m$ represents a control force vector, whose component indicates a control force produced by a control system; $\boldsymbol{M}$, $\boldsymbol{C}$ and $\boldsymbol{K} \in \mathbb{R}^{n\times n}$ represent mass, damping and stiffness matrices of the model, respectively; $\boldsymbol{V} \in \mathbb{R}^n$ indicates the DOFs where the seismic excitation acts; $\boldsymbol{U} \in \mathbb{R}^{n\times m}$ represents the DOFs where the control forces act; and $\mathbb{R}^p$ and $\mathbb{R}^{p\times r}$ indicate a p-dimensional vector and a $p \times r$ dimensional matrix.

The structural dynamics with the controlled forces provided by the VSE and so on is expressed as:

$$\boldsymbol{M}\boldsymbol{y}''(t) + \boldsymbol{C}\boldsymbol{y}'(t) + \boldsymbol{K}\boldsymbol{y}(t) + \boldsymbol{U}\boldsymbol{u}(t) = -\boldsymbol{M}\boldsymbol{V}w(t). \tag{2.2}$$

Since we can convert the $\boldsymbol{U}\boldsymbol{u}(t)$ term to the right-hand side and change the sign of $\boldsymbol{u}(t)$, Eqs. (2.1) and (2.2) are mathematically equivalent. Thus, the case expressed by Eq. (2.1) is mainly analyzed in this paper.

Example 2.1: model-S

As an example, consider an SDOF model whose mass, damping and stiffness are m, c and k, respectively. Then, letting natural circular frequency ω and a damping factor h be $\omega = \sqrt{k/m}$ and $h = c/(2\omega)$, respectively, the structural dynamics is express by:

$$y''(t) + 2h\omega y'(t) + \omega^2 y(t) = -w(t) + m^{-1}u(t). \tag{2.3}$$

Example 2.2: model-M

As an example of an MDOF model, let us assume a 3DOF model that comprises three masses and interstory springs, as shown in Fig. 2. To let the control forces act between DOF, $\boldsymbol{U}\boldsymbol{u}(t)$ is defined by:

$$\boldsymbol{U}\boldsymbol{u}(t) = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{Bmatrix}. \tag{2.4}$$

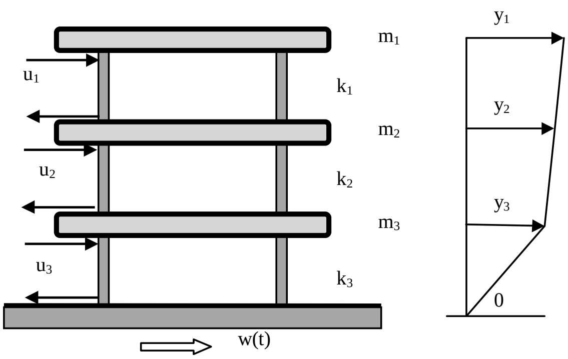

Fig. 2. MDOF model with control force.

2.3. State equation

Eq. (2.1) can be converted to a so-called state equation:

$$\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{D}w(t) + \boldsymbol{B}\boldsymbol{u}(t), \tag{2.5}$$

where

$$\boldsymbol{x}(t) = \begin{Bmatrix} \boldsymbol{y}'(t) \\ \boldsymbol{y}(t) \end{Bmatrix}, \qquad \boldsymbol{A} = \begin{bmatrix} -\boldsymbol{M}^{-1}\boldsymbol{C} & -\boldsymbol{M}^{-1}\boldsymbol{K} \\ \boldsymbol{I} & \boldsymbol{0} \end{bmatrix}, \qquad \boldsymbol{D} = \begin{bmatrix} -\boldsymbol{V} \\ \boldsymbol{0} \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} \boldsymbol{M}^{-1}\boldsymbol{U} \\ \boldsymbol{0} \end{bmatrix},$$

i.e., $\boldsymbol{x}(t) \in \mathbb{R}^{2n}$, $\boldsymbol{A} \in \mathbb{R}^{2n\times 2n}$, $\boldsymbol{D} \in \mathbb{R}^{2n}$ and $\boldsymbol{B} \in \mathbb{R}^{2n\times m}$.

When all the eigenvalues of $\boldsymbol{A}$ in Eq. (2.5) are less than 0.0, which is called a Hurwitz matrix, we have:

$$\boldsymbol{\Xi}(t) = \exp(t\boldsymbol{A}) = \boldsymbol{I} + t\boldsymbol{A} + t^2\boldsymbol{A}^2/2 + \cdots \tag{2.6}$$

Then, structural responses at t are obtained for the initial condition $\boldsymbol{x}(t_0)$ as:

$$\boldsymbol{x}(t) = \boldsymbol{\Xi}(t-t_0)\boldsymbol{x}(t_0) + (\boldsymbol{\Xi}^*\boldsymbol{D}w)(t,t_0) + (\boldsymbol{\Xi}^*\boldsymbol{B}\boldsymbol{u})(t,t_0), \tag{2.7}$$

where $*$ indicates the convolution:

$$(\boldsymbol{\Xi}^*\boldsymbol{f})(t,t_0) = \int_{t_0}^{t} \boldsymbol{\Xi}(\tau - t_0)\boldsymbol{f}(t-\tau)\,\mathrm{d}\tau = \int_{t_0}^{t} \boldsymbol{\Xi}(t-\tau)\boldsymbol{f}(\tau - t_0)\,\mathrm{d}\tau. \tag{2.8}$$

Eq. (2.5) can be discretized at $t = r\Delta t$, assuming sampling time Δt:

$$\boldsymbol{x}(r+1) = \hat{\boldsymbol{A}}\boldsymbol{x}(r) + \hat{\boldsymbol{D}}w(r) + \hat{\boldsymbol{B}}\boldsymbol{u}(r), \tag{2.9}$$

where $\hat{\boldsymbol{A}} = \boldsymbol{\Xi}(\Delta t)$, $\hat{\boldsymbol{D}} = (\boldsymbol{\Xi}^*\boldsymbol{D})(\Delta t, 0)$, $\hat{\boldsymbol{B}} = (\boldsymbol{\Xi}^*\boldsymbol{B})(\Delta t, 0)$; and $\boldsymbol{x}(r)$, $\boldsymbol{u}(r)$ and $w(r)$ indicate values at step r.

Example 2.3: discrete SDOF model

For the SDOF model expressed by Eq. (2.3), we have:

$$\boldsymbol{A} = \begin{bmatrix} -m^{-1}c & -m^{-1}k \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2h\omega & -\omega^2 \\ 1 & 0 \end{bmatrix}, \qquad \boldsymbol{D} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} m^{-1} \\ 0 \end{bmatrix}.$$

Let us assume:

$$\lambda_1, \lambda_2 = \omega\left(-h \pm \mathrm{i}\sqrt{1-h^2}\right), \quad \Lambda_1 = \exp(\lambda_1\Delta t), \quad \Lambda_2 = \exp(\lambda_2\Delta t), \quad \text{and } \boldsymbol{\Phi} = \frac{1}{\sqrt{2}}\begin{Bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{Bmatrix}.$$

Then,

$$\hat{\boldsymbol{A}} = \exp(\boldsymbol{A}\Delta t) = \boldsymbol{\Phi}\,\mathrm{diag}\{\Lambda_1 \quad \Lambda_2\}\boldsymbol{\Phi}^{-1} = \frac{1}{\lambda_1 - \lambda_2}\begin{bmatrix} \lambda_1\Lambda_1 - \lambda_2\Lambda_2 & \lambda_1\lambda_2(\Lambda_2 - \Lambda_1) \\ \Lambda_1 - \Lambda_2 & \lambda_1\Lambda_2 - \lambda_2\Lambda_1 \end{bmatrix},$$

$$(\boldsymbol{\Xi}^*\boldsymbol{I})(\Delta t, 0) = \frac{1}{\lambda_1\lambda(\lambda_1 - \lambda_2)}\begin{bmatrix} \lambda_1\lambda_2(\Lambda_1 - \Lambda_2) & \lambda_1\lambda_2\{\lambda_2(1-\Lambda_1) - \lambda_1(1-\Lambda_2)\} \\ \lambda_1(1-\Lambda_2) - \lambda_2(1-\Lambda_1) & -\lambda_1^2(1-\Lambda_2) + \lambda_2^2(1-\Lambda_1) \end{bmatrix},$$

$$\hat{\boldsymbol{D}} = \frac{1}{\lambda_1\lambda_2(\lambda_1 - \lambda_2)}\begin{bmatrix} \lambda_1\lambda_2(\Lambda_1 - \Lambda_2) \\ \lambda_1(1-\Lambda_2) - \lambda_2(1-\Lambda_1) \end{bmatrix}, \qquad \hat{\boldsymbol{B}} = -m^{-1}\hat{\boldsymbol{D}}. \tag{2.10}$$

2.4. Mode decomposition

Let us estimate control force influence on the ith mode, assuming that $\boldsymbol{M}^{-1}\boldsymbol{C}$ can be decomposed by eigenvector $\boldsymbol{\varphi}_i$ with $\boldsymbol{\varphi}_i^{\mathrm{T}}\boldsymbol{\varphi}_i = 1$. By multiplying $\boldsymbol{\varphi}_i^{\mathrm{T}}\boldsymbol{M}^{-1}$ to Eq. (2.1):

$$\boldsymbol{\varphi}_i^{\mathrm{T}}\boldsymbol{y}''(t) + \boldsymbol{\varphi}_i^{\mathrm{T}}\boldsymbol{M}^{-1}\boldsymbol{C}\boldsymbol{\varphi}_i\boldsymbol{\varphi}_i^{\mathrm{T}}\boldsymbol{y}'(t) + \boldsymbol{\varphi}_i^{\mathrm{T}}\boldsymbol{M}^{-1}\boldsymbol{K}\boldsymbol{\varphi}_i\boldsymbol{\varphi}_i^{\mathrm{T}}\boldsymbol{y}(t) = -\boldsymbol{\varphi}_i^{\mathrm{T}}\boldsymbol{V}w(t) + \boldsymbol{\varphi}_i^{\mathrm{T}}\boldsymbol{M}^{-1}\boldsymbol{U}\boldsymbol{u}(t). \tag{2.11}$$

By expressing $\boldsymbol{U} = \{..\boldsymbol{U}_j..\}$, the ith mode $y_i^*(t) = \boldsymbol{\varphi}_i^{\mathrm{T}}\boldsymbol{y}(t)$ is governed by:

$$y_i^{*\prime\prime}(t) + 2h_i\omega_i y_i^{*\prime}(t) + \omega_i^2 y_i^*(t) = -\beta_i w(t) + \sum_j \alpha_{ij}u_j(t), \tag{2.12}$$

where the ith natural circular frequency: $\omega_i^2 = \boldsymbol{\varphi}_i^{\mathrm{T}}\boldsymbol{M}^{-1}\boldsymbol{K}\boldsymbol{\varphi}_i$,

the ith modal damping: $h_i = \boldsymbol{\varphi}_i^{\mathrm{T}}\boldsymbol{M}^{-1}\boldsymbol{C}\boldsymbol{\varphi}_i/2\omega_i$,
influence factor of the jth control force on the ith mode: $\alpha_{ij} = \boldsymbol{\varphi}_i^{\mathrm{T}}\boldsymbol{M}^{-1}\boldsymbol{U}_j$,
participation factor of the excitation to the ith mode: $\beta_i = \boldsymbol{\varphi}_i^{\mathrm{T}}\boldsymbol{V}$.

Thus, we should note that control force locations largely influence control effects, regardless of control laws.

By multiplying $\{\boldsymbol{\varphi}_i^{\mathrm{T}}\boldsymbol{\varphi}_i^{\mathrm{T}}\}$ to Eq. (2.5), the state equation is decomposed to:

$$\boldsymbol{x}_i''(t) = \{\boldsymbol{\varphi}_i^{\mathrm{T}}\boldsymbol{\varphi}_i^{\mathrm{T}}\}\boldsymbol{x}(t) = \begin{Bmatrix} y_i^{*\prime}(t) \\ y_i^*(t) \end{Bmatrix}, \tag{2.13}$$

where

$$\boldsymbol{x}_i^*(t) = \{\boldsymbol{\varphi}_i^{\mathrm{T}}\boldsymbol{\varphi}_i^{\mathrm{T}}\}\boldsymbol{x}(t) = \begin{Bmatrix} y_i^{\prime *}(t) \\ y_i^*(t) \end{Bmatrix}, \qquad \boldsymbol{A}_i^* = \{\boldsymbol{\varphi}_i^{\mathrm{T}}\boldsymbol{\varphi}_i^{\mathrm{T}}\}\boldsymbol{A}\begin{Bmatrix} \boldsymbol{\varphi}_i \\ \boldsymbol{\varphi}_i \end{Bmatrix} = \begin{bmatrix} -2h_i\omega_i & -\omega_i^2 \\ 1 & 0 \end{bmatrix},$$

$$\boldsymbol{D}_i^* = \{\boldsymbol{\varphi}_i^{\mathrm{T}}\boldsymbol{\varphi}_i^{\mathrm{T}}\}\boldsymbol{D} = \begin{Bmatrix} -\beta_i \\ 0 \end{Bmatrix}, \qquad \boldsymbol{B}_i^* = \{\boldsymbol{\varphi}_i^{\mathrm{T}}\boldsymbol{\varphi}_i^{\mathrm{T}}\}\boldsymbol{B} = \begin{bmatrix} \alpha_{i1} & .. & \alpha_{ij} & .. & \alpha_{im} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The contents of $\boldsymbol{B}_i^*$ imply that acceleration, i.e., velocity increment, has direct influence of the control force, while displacement is controlled via velocity.

Eq. (2.13) can be converted to a pair of complex conjugates $z_i^*(t)$ and $\bar{z}_i^*(t)$, using $\boldsymbol{\Phi} = \{\cdot\cdot\boldsymbol{\varphi}_i\cdot\cdot\}$, $\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\Phi}^{\mathrm{T}} = \boldsymbol{I}$,

$$\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{x}_i^{*\prime}(t) = \boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{A}_i^*\boldsymbol{\Phi}\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{x}_i^*(t) + \boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{D}_i^*w(t) + \boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{B}_i^*\boldsymbol{u}(t). \tag{2.14}$$

Letting $\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{x}_i^*(t) = \{z_i^*(t)\bar{z}_i^*(t)\}^{\mathrm{T}}$ and $\mathrm{i} = \sqrt{-1}$

$$z_i^*(t), \bar{z}_i^*(t) = \frac{\omega_i}{2}y_i^*(t) \pm \mathrm{i}\frac{1}{2\omega\sqrt{1-h_i^2}}(y_i^{*\prime}(t) + h_i\omega_i y_i^*(t)). \tag{2.15}$$

Hence,

$$z_i^{*\prime}(t) = \omega_i\left(-h_i + \mathrm{i}\sqrt{1-h_i^2}\right)z_i^*(t) + \mathrm{i}\frac{1}{2\omega\sqrt{1-h_i^2}}\left(-\beta_i w(t) + \sum_j \alpha_{ij}u_j(t)\right). \tag{2.16}$$

Thus, the control force acts on the response z_i^* via an imaginary unit.

2.5. Energy balance

By multiplying Eq. (2.1) by $\boldsymbol{y}'(t)^{\mathrm{T}}$ and integrating the result from t_0 to t_1, the energy balance from time t_0 to t_1 is obtained:

$$\left[\int_{t_0}^{t_1}\left(\boldsymbol{y}'(t)^{\mathrm{T}}\boldsymbol{M}\boldsymbol{y}''(t)+\boldsymbol{y}'(t)^{\mathrm{T}}\boldsymbol{K}\boldsymbol{y}(t)\right)\mathrm{d}t\right]+\left[\int_{t_0}^{t_1}\boldsymbol{y}'(t)^{\mathrm{T}}\boldsymbol{C}\boldsymbol{y}'(t)\,\mathrm{d}t\right]$$
$$=\left[\int_{t_0}^{t_1}-\boldsymbol{y}'(t)^{\mathrm{T}}\boldsymbol{M}\boldsymbol{V}w(t)\,\mathrm{d}t\right]+\left[\int_{t_0}^{t_1}\boldsymbol{y}'(t)^{\mathrm{T}}\boldsymbol{U}\boldsymbol{u}(t)\,\mathrm{d}t\right]. \tag{2.17}$$

The terms in brackets are called the vibration, damping, excitation energy and control force energy from t_0 to t_1, respectively.

And, by applying Leipniz's rule to the first term, we have:

$$\left[\frac{1}{2}\boldsymbol{y}'(t_1)^{\mathrm{T}}\boldsymbol{M}\boldsymbol{y}'(t_1)+\frac{1}{2}\boldsymbol{y}(t_1)^{\mathrm{T}}\boldsymbol{K}\boldsymbol{y}(t_1)\right]+\left[\int_{t_0}^{t_1}\boldsymbol{y}'(t)^{\mathrm{T}}\boldsymbol{C}\boldsymbol{y}'(t)\,\mathrm{d}t\right]+\left[-\int_{t_0}^{t_1}\boldsymbol{y}'(t)^{\mathrm{T}}\boldsymbol{U}\boldsymbol{u}(t)\,\mathrm{d}t\right]$$
$$=\left[\int_{t_0}^{t_1}-\boldsymbol{y}'(t)^{\mathrm{T}}\boldsymbol{M}\boldsymbol{V}w(t)\,\mathrm{d}t\right]+\left[\frac{1}{2}\boldsymbol{y}'(t_0)^{\mathrm{T}}\boldsymbol{M}\boldsymbol{y}'(t_0)+\frac{1}{2}\boldsymbol{y}(t_0)\boldsymbol{K}\boldsymbol{y}(t_0)\right]. \tag{2.18}$$

The right-hand side indicates the input energy from t_0 to t_1.

To obtain the same results from the state equation, let:

$$\boldsymbol{P}=\begin{bmatrix}\boldsymbol{M} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{K}\end{bmatrix}. \tag{2.19}$$

Then, by multiplying Eq. (2.5) by $\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}$ and integrating the result from time t_0 to t_1, we have:

$$\int_{t_0}^{t_1}\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}'(t)\,\mathrm{d}t=\int_{t_0}^{t_1}\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{A}\boldsymbol{x}(t)\,\mathrm{d}t+\int_{t_0}^{t_1}\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{D}w(t)\,\mathrm{d}t+\int_{t_0}^{t_1}\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{B}\boldsymbol{u}(t)\,\mathrm{d}t, \tag{2.20}$$

where:

$$\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}'(t)=\{\boldsymbol{y}'(t)^{\mathrm{T}}\boldsymbol{y}(t)\}\begin{bmatrix}\boldsymbol{M} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{K}\end{bmatrix}\begin{Bmatrix}\boldsymbol{y}''(t)\\ \boldsymbol{y}'(t)\end{Bmatrix}=\boldsymbol{y}'(t)^{\mathrm{T}}\boldsymbol{M}\boldsymbol{y}''(t)+\boldsymbol{y}(t)^{\mathrm{T}}\boldsymbol{K}\boldsymbol{y}'(t),$$

$$\int_{t_0}^{t_1}\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}'(t)\,\mathrm{d}t=\frac{1}{2}\boldsymbol{y}'(t_1)^{\mathrm{T}}\boldsymbol{M}\boldsymbol{y}'(t_1)+\frac{1}{2}\boldsymbol{y}(t_1)^{\mathrm{T}}\boldsymbol{K}\boldsymbol{y}(t_1)-\frac{1}{2}\boldsymbol{y}'(t_0)^{\mathrm{T}}\boldsymbol{M}\boldsymbol{y}'(t_0)-\frac{1}{2}\boldsymbol{y}(t_0)^{\mathrm{T}}\boldsymbol{K}\boldsymbol{y}(t_0),$$

$$\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{A}\boldsymbol{x}(t)=\{\boldsymbol{y}'(t)^{\mathrm{T}}\boldsymbol{y}(t)\}\begin{bmatrix}\boldsymbol{M} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{K}\end{bmatrix}\begin{bmatrix}-\boldsymbol{M}^{-1}\boldsymbol{C} & -\boldsymbol{M}^{-1}\boldsymbol{K}\\ \boldsymbol{I} & \boldsymbol{0}\end{bmatrix}\begin{Bmatrix}\boldsymbol{y}'(t)\\ \boldsymbol{y}(t)\end{Bmatrix}=-\boldsymbol{y}'(t)^{\mathrm{T}}\boldsymbol{C}\boldsymbol{y}'(t),$$

$$\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{D}w(t)=\{\boldsymbol{y}'(t)^{\mathrm{T}}\boldsymbol{y}(t)\}\begin{bmatrix}\boldsymbol{M} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{K}\end{bmatrix}\begin{Bmatrix}-\boldsymbol{V}\\ \boldsymbol{0}\end{Bmatrix}w(t)=-\boldsymbol{y}'(t)^{\mathrm{T}}\boldsymbol{M}\boldsymbol{V}w(t),$$

$$\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{B}\boldsymbol{u}(t)=\{\boldsymbol{y}'(t)^{\mathrm{T}}\boldsymbol{y}(t)\}\begin{bmatrix}\boldsymbol{M} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{K}\end{bmatrix}\begin{bmatrix}\boldsymbol{M}^{-1}\boldsymbol{U}\\ \boldsymbol{0}\end{bmatrix}\boldsymbol{u}(t)=\boldsymbol{y}'(t)^{\mathrm{T}}\boldsymbol{U}\boldsymbol{u}(t). \tag{2.21}$$

Thus, the term in the left side of Eq. (2.20) indicates the difference between the vibration energies at t_1 and t_0. On the other hand, each term in the right-hand side represents the integration of the damping, excitation, and control force energy, respectively.

2.6. Structure with Maxwell elements

In the foregoing, control forces even by a semi-active system are assumed to act directly on a structure. However, a damping element must often be installed in a structure via an ASE, thus composing a Maxwell element, as shown in Fig. 3. Even if it is directly attached to structural elements, we should consider its

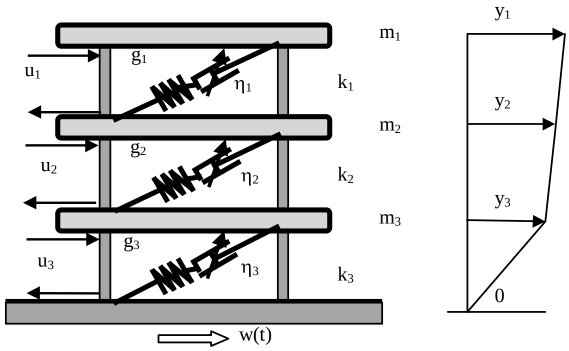

Fig. 3. MDOF model with a Maxwell element.

stiffness. It is thus important to examine the basic performance of a structural model with Maxwell elements.

Let us assume an SDOF structure with a Maxwell element. The values g and η represent the stiffness of an ASE and the damping factor of a damping element.

$$my''(t) + cy'(t) + ky(t) + u(t) = -mw(t), u'(t) = gy'(t) - (g/\eta)u(t). \tag{2.22}$$

where $u(t)$ indicates a force occurring in a Maxwell element.

By a state equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{Bmatrix} y'(t) \\ y(t) \\ u(t) \end{Bmatrix} = \begin{bmatrix} -c/m & -k/m & -1/m \\ 1 & 0 & 0 \\ g & 0 & -g/\eta \end{bmatrix}\begin{Bmatrix} y'(t) \\ y(t) \\ u(t) \end{Bmatrix} + \begin{Bmatrix} -1 \\ 0 \\ 0 \end{Bmatrix} w(t). \tag{2.23}$$

The eigenvalues of the matrix in the above provide a natural circular frequency λ. Then,

$$\lambda^3 + \left(\frac{c}{m} + \frac{g}{\eta}\right)\lambda^2 + \left(\frac{g}{\eta}\frac{c}{m} + \frac{1}{m}g + \frac{k}{m}\right)\lambda + \frac{g}{\eta}\frac{k}{m} = 0. \tag{2.24}$$

Assuming $a = (c/m) + (g/\eta)$, $b = (g/\eta)(c/m) + (1/m)g + (k/m)$ and $d = (g/\eta)(1/m)$, solutions are

$$\lambda_1, \lambda_2 = \frac{1}{2}(e+f) - \frac{a}{3} \pm \mathrm{i}\frac{\sqrt{3}}{2}(e-f), \qquad \lambda_3 = e + f - \frac{a}{3}, \tag{2.25}$$

where

$$e = \alpha^{1/3}, \qquad f = \beta^{1/3}, \qquad \alpha, \beta = \left(-q \pm \sqrt{q^2 + 4p^3}\right)/2, \qquad p = (3b - a^2)/9,$$
$$q = (2b^3 - 9bc + 27d).$$

The most important circular frequency ω_1, natural period T_1, damping factor h_1 are given by:

$$\omega_1 = |\lambda_1|, \qquad T_1 = 2\pi/\mathrm{imag}(\lambda_1), \qquad h_1 = \mathrm{real}(\lambda_1)/|\lambda_1|. \tag{2.26}$$

Example 2.4 Fig. 4 shows T_1 and h_1 for $m = 1$, $c = 0.02\pi$, $k = 4\pi^2$, $g = 4\pi^2$, $8\pi^2$ and $16\pi^2$, and $\eta = 0.5$–2.0. The structural natural periods as well as the damping factors change, depending on the damping coefficient in a Maxwell element. Thus, we have the optimum damping coefficient for achieving the highest damping effect.

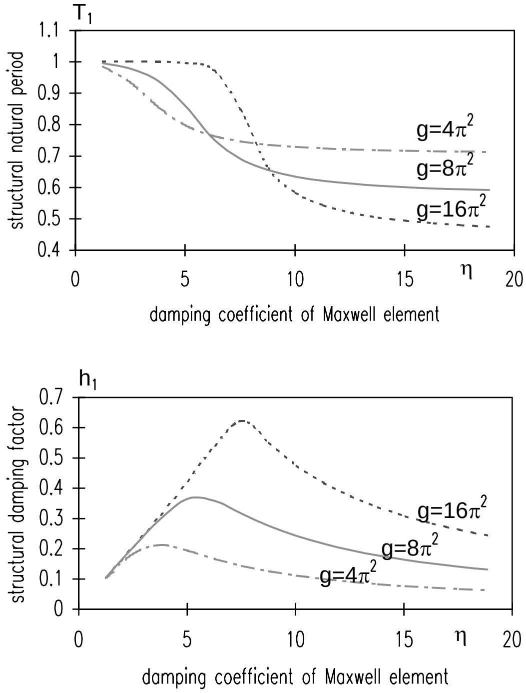

Fig. 4. Natural period and damping factor of a structure with a Maxwell element.

To examine general natures, we should scale Eq. (2.22), assuming:

$$c = 2hm\omega, \quad k = m\omega^2, \quad g = m\omega^2 g^*, \quad \eta = m\omega\eta^*, \quad u(t) = m\omega^2 u^*(\tau), \quad \tau = \omega t \quad (\ )' = \omega(\ )^{\cdot}.$$

That is,

$$\ddot{y}(\tau) + 2hy(\tau) + \omega^2 y(\tau) + u^*(\tau) = -\frac{1}{\omega^2}w(t), \tag{2.27}$$

$$\dot{u}(\tau) = g^*\dot{y}(\tau) - (g^*/\eta^*)u^*(\tau). \tag{2.28}$$

Thus, g and η have influence of ω^2 and ω, respectively.

In the same way as for an SDOF model, let us assume an MDOF structural model equipped with VDE via ASEs, which compose NMW elements. By assuming that g_j and η_j represent the stiffness of the jth ASE and the damping coefficient of jth damping element, the structural dynamics are expressed by:

$$\boldsymbol{M}\boldsymbol{y}''(t) + \boldsymbol{C}\boldsymbol{y}'(t) + \boldsymbol{K}\boldsymbol{y}(t) + \boldsymbol{U}\boldsymbol{u}(t) = -\boldsymbol{M}\boldsymbol{V}w(t), \tag{2.29}$$

$$\boldsymbol{u}'(t) = \boldsymbol{G}\boldsymbol{U}^{\mathrm{T}}\boldsymbol{y}'(t) - \boldsymbol{E}(t)\boldsymbol{u}(t), \tag{2.30}$$

where $\boldsymbol{G} = \mathrm{diag}\{g_j\}$, $\boldsymbol{E}(t) = \mathrm{diag}\{g_j/\eta_j(t)\}$, $\boldsymbol{U}$ indicates the DOFs where Maxwell elements are attached; and diag{ } composes a diagonal matrix of a vector { }.

To obtain an energy equation, let us multiply Eq. (2.29) by $\boldsymbol{y}'(t)^{\mathrm{T}}$ and Eq. (2.30) by $\boldsymbol{u}(t)^{\mathrm{T}}\boldsymbol{G}^{-1}$:

$$\boldsymbol{y}'(t)^{\mathrm{T}}\boldsymbol{M}\boldsymbol{y}''(t) + \boldsymbol{y}'(t)^{\mathrm{T}}\boldsymbol{C}\boldsymbol{y}'(t) + \boldsymbol{y}'(t)^{\mathrm{T}}\boldsymbol{K}\boldsymbol{y}(t) + \boldsymbol{y}'(t)^{\mathrm{T}}\boldsymbol{U}\boldsymbol{u}(t) = -\boldsymbol{y}'(t)^{\mathrm{T}}\boldsymbol{M}\boldsymbol{V}w(t), \tag{2.31}$$

$$\boldsymbol{u}(t)^{\mathrm{T}}\boldsymbol{G}^{-1}\boldsymbol{u}'(t) = \boldsymbol{u}(t)^{\mathrm{T}}\boldsymbol{U}^{\mathrm{T}}\boldsymbol{y}'(t) - \boldsymbol{u}(t)^{\mathrm{T}}\boldsymbol{G}^{-1}\boldsymbol{E}(t)\boldsymbol{u}(t). \tag{2.32}$$

By substituting Eq. (2.32) into Eq. (2.31), and integrating the result from t_0 to t_1, we have:

$$\frac{1}{2}\left[\boldsymbol{y}'(t_1)^{\mathrm{T}}\boldsymbol{M}\boldsymbol{y}'(t_1)+\boldsymbol{y}(t_1)^{\mathrm{T}}\boldsymbol{K}\boldsymbol{y}(t_1)+\boldsymbol{u}(t_1)^{\mathrm{T}}\boldsymbol{G}^{-1}\boldsymbol{u}(t_1)\right]+\int_{t_0}^{t_1}\boldsymbol{y}'(t)^{\mathrm{T}}\boldsymbol{C}\boldsymbol{y}'(t)\,\mathrm{d}t+\int_{t_0}^{t_1}\boldsymbol{u}(t)^{\mathrm{T}}\boldsymbol{G}^{-1}\boldsymbol{E}(t)\boldsymbol{u}(t)\,\mathrm{d}t$$
$$=\left[\int_{t_0}^{t_1}-\boldsymbol{y}'(t)^{\mathrm{T}}\boldsymbol{M}\boldsymbol{V}w(t)\,\mathrm{d}t\right]+\frac{1}{2}\left[\boldsymbol{y}'(t_0)^{\mathrm{T}}\boldsymbol{M}\boldsymbol{y}'(t_0)+\boldsymbol{y}(t_0)\boldsymbol{K}\boldsymbol{y}(t_0)+\boldsymbol{u}(t_0)\boldsymbol{G}^{-1}\boldsymbol{u}(t_0)\right]. \tag{2.33}$$

Thus, as far as $\boldsymbol{G} > 0$ and $\boldsymbol{E}(t) > 0$, the energy is dissipated by NMW elements even if $\boldsymbol{E}(t)$ is variable.

3. Linear feedback control laws

3.1. Linear state feedback laws

Let us consider a control force vector produced by linear state FB laws as:

$$\boldsymbol{u}(t)=-\boldsymbol{G}(t)\boldsymbol{x}(t)+\boldsymbol{g}(t)w(t)=-\boldsymbol{G}_{\mathrm{v}}(t)\boldsymbol{y}'(t)-\boldsymbol{G}_{\mathrm{d}}(t)\boldsymbol{y}(t)+\boldsymbol{g}(t)w(t), \tag{3.1}$$

where $\boldsymbol{G}(t)\in\mathbb{R}^{m\times 2n}$, $\boldsymbol{G}_{\mathrm{v}}(t)\in\mathbb{R}^{m\times n}$, $\boldsymbol{G}_{\mathrm{d}}(t)\in\mathbb{R}^{m\times n}$ and $\boldsymbol{g}(t)\in\mathbb{R}^{m}$ represent the gains to state values, velocity, displacement and excitation, respectively.

By putting Eq. (3.1) into Eq. (2.1), we obtain:

$$\boldsymbol{M}\boldsymbol{y}''(t)+(\boldsymbol{C}+\boldsymbol{U}\boldsymbol{G}_{\mathrm{v}}(t))\boldsymbol{y}'(t)+(\boldsymbol{K}+\boldsymbol{U}\boldsymbol{G}_{\mathrm{d}}(t))\boldsymbol{y}(t)=-(\boldsymbol{M}\boldsymbol{V}-\boldsymbol{U}\boldsymbol{g}(t))w(t). \tag{3.2}$$

The energy balance from t_0 to t_1 is:

$$\left[\frac{1}{2}\boldsymbol{y}'(t_1)^{\mathrm{T}}\boldsymbol{M}\boldsymbol{y}'(t_1)+\frac{1}{2}\boldsymbol{y}(t_1)^{\mathrm{T}}\boldsymbol{K}\boldsymbol{y}(t_1)\right]+\left[\int_{t_0}^{t_1}\boldsymbol{y}'(t)^{\mathrm{T}}\boldsymbol{C}\boldsymbol{y}'(t)\,\mathrm{d}t\right]+\left\{\int_{t_0}^{t_1}\boldsymbol{y}'(t)^{\mathrm{T}}\boldsymbol{U}\boldsymbol{G}_{\mathrm{v}}(t)\boldsymbol{y}'(t)\,\mathrm{d}t\right\}$$
$$+\left\{\int_{t_0}^{t_1}\boldsymbol{y}'(t)^{\mathrm{T}}\boldsymbol{U}\boldsymbol{G}_{\mathrm{d}}(t)\boldsymbol{y}(t)\,\mathrm{d}t\right\}+\left\{\int_{t_0}^{t_1}-\boldsymbol{y}'(t)^{\mathrm{T}}\boldsymbol{U}\boldsymbol{g}(t)w(t)\,\mathrm{d}t\right\}$$
$$=\left[\int_{t_0}^{t_1}-\boldsymbol{y}'(t)^{\mathrm{T}}\boldsymbol{M}\boldsymbol{V}w(t)\,\mathrm{d}t\right]+\left[\frac{1}{2}\boldsymbol{y}'(t_0)^{\mathrm{T}}\boldsymbol{M}\boldsymbol{y}'(t_0)+\frac{1}{2}\boldsymbol{y}(t_0)^{\mathrm{T}}\boldsymbol{K}\boldsymbol{y}(t_0)\right]. \tag{3.3}$$

The terms in braces are called the control force energies produced by the velocity, deformation and excitation FB, respectively. Then, assuming certain gains, let us analyze linear FB control laws.

3.2. Velocity feedback laws

Let $\boldsymbol{U}\boldsymbol{G}_{\mathrm{v}}(t)$ be positive definite. Then, $\boldsymbol{y}'(t)^{\mathrm{T}}\boldsymbol{U}\boldsymbol{G}_{\mathrm{v}}(t)\boldsymbol{y}'(t)$ is always nonnegative, so that the control force continuously dissipates vibration energy. Because of their robustness and time independency, the velocity FB laws are widely used.

3.3. Deformation feedback laws

From the viewpoint of energy dissipation, let us find an appropriate $\boldsymbol{G}_{\mathrm{d}}(t)$. If we assume that $\boldsymbol{U}\boldsymbol{G}_{\mathrm{d}}(t)=(\boldsymbol{U}\boldsymbol{G}_{\mathrm{d}}(t))^{\mathrm{T}}$, the problem is easier. Such gain is defined by:

$$\boldsymbol{G}_{\mathrm{d}}(t)=\boldsymbol{P}_{\mathrm{d}}(t)\boldsymbol{U}^{\mathrm{T}}, \tag{3.4}$$

where $\boldsymbol{P}_{\mathrm{d}}(t)=\boldsymbol{P}_{\mathrm{d}}(t)^{\mathrm{T}}\in\mathbb{R}^{n\times n}$, i.e., deformations between the locations where the control forces act are fed back.

Then,

$$\int_{t_0}^{t_1} \boldsymbol{y}'(t)^{\mathrm{T}} \boldsymbol{U}\boldsymbol{G}_{\mathrm{d}}(t)\boldsymbol{y}(t)\,\mathrm{d}t = \left[\frac{1}{2}\boldsymbol{y}(t)^{\mathrm{T}}\boldsymbol{U}\boldsymbol{P}_{\mathrm{d}}(t)\boldsymbol{U}^{\mathrm{T}}\boldsymbol{y}(t)\right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \frac{1}{2}\boldsymbol{y}(t)^{\mathrm{T}}\boldsymbol{U}\boldsymbol{P}'_{\mathrm{d}}(t)\boldsymbol{U}^{\mathrm{T}}\boldsymbol{y}(t)\,\mathrm{d}t. \tag{3.5}$$

The second term in the right-hand side with $\boldsymbol{P}'_{\mathrm{d}}(t) \leqslant 0$ suggests energy dissipation from t_0 to t_1 while that with $\boldsymbol{P}'_{\mathrm{d}}(t) > 0$ works negatively. Thus, for energy dissipation, softening is preferable to hardening, or the change with $\boldsymbol{P}'_{\mathrm{d}}(t) > 0$ should be done quickly or while $\boldsymbol{y}(t)$ is small.

If $\boldsymbol{P}'_{\mathrm{d}}(t) > 0$, a quick change $\Delta\boldsymbol{P}_{\mathrm{d}} = \boldsymbol{P}_{\mathrm{d}}(t_a + \Delta t) - \boldsymbol{P}_{\mathrm{d}}(t_a)$ at t_a will keep deformation, i.e., $\boldsymbol{y}(t_a + \Delta t) \sim \boldsymbol{y}(t_a)$, so that:

$$\frac{1}{2}\boldsymbol{y}(t_a+\Delta t)^{\mathrm{T}}\boldsymbol{U}\boldsymbol{P}_{\mathrm{d}}(t_a+\Delta t)\boldsymbol{U}^{\mathrm{T}}\boldsymbol{y}(t_a+\Delta t) - \frac{1}{2}\boldsymbol{y}(t_a)^{\mathrm{T}}\boldsymbol{U}\boldsymbol{P}_{\mathrm{d}}(t_a)\boldsymbol{U}^{\mathrm{T}}\boldsymbol{y}(t_a) \sim \frac{1}{2}\boldsymbol{y}(t_a)^{\mathrm{T}}\boldsymbol{U}\Delta\boldsymbol{P}_{\mathrm{d}}\boldsymbol{U}^{\mathrm{T}}\boldsymbol{y}(t_a),$$

$$\begin{aligned} -\int_{t_a}^{t_a+\Delta t} \frac{1}{2}\boldsymbol{y}(t)^{\mathrm{T}}\boldsymbol{U}\boldsymbol{P}'_{\mathrm{d}}(t)\boldsymbol{U}^{\mathrm{T}}\boldsymbol{y}(t)\,\mathrm{d}t &= -\boldsymbol{y}(t_a)^{\mathrm{T}}\boldsymbol{U}\left[\int_{t_a}^{t_a+\Delta t} \frac{1}{2\Delta t}\,\Delta\boldsymbol{P}_{\mathrm{d}}\,\mathrm{d}t\right]\boldsymbol{U}^{\mathrm{T}}\boldsymbol{y}(t_a) \\ &\sim -\frac{1}{2}\boldsymbol{y}(t_a)^{\mathrm{T}}\boldsymbol{U}\Delta\boldsymbol{P}_{\mathrm{d}}\boldsymbol{U}^{\mathrm{T}}\boldsymbol{y}(t_a). \end{aligned} \tag{3.6}$$

Thus, the second term on the right-hand side of Eq. (3.5) can be recovered by the first term. Then, for the time interval from t_1 to t_2 which involves the quick change at t_a,

$$\int_{t_0}^{t_1} \boldsymbol{y}'(t)^{\mathrm{T}}\boldsymbol{U}\boldsymbol{P}_{\mathrm{d}}(t)\boldsymbol{U}^{\mathrm{T}}\boldsymbol{y}(t)\,\mathrm{d}t = \int_{t_0}^{t_a} + \int_{t_a}^{t_a+\Delta t} + \int_{t_a+\Delta t}^{t_1} \sim \int_{t_0}^{t_a} + \int_{t_a+\Delta t}^{t_1}. \tag{3.7}$$

Thus, a control force with $\boldsymbol{P}'_{\mathrm{d}}(t) < 0$ for most of the time but with $\boldsymbol{P}'_{\mathrm{d}}(t) > 0$ for limited periods can dissipate energy from t_0 to t_1. In fact, a hysteretic force by an elastoplastic constitutive law satisfies this condition, so that it can dissipate energy.

We can anticipate another effect by deformation FB, i.e., a dissonant effect. As shown by Eq. (3.2), the deformation FB has influence as if it can change the structural natural period. Such influences are estimated in Section 3.5, assuming constant gains.

3.4. Excitation feedback laws

Let us assume that $\boldsymbol{U}\boldsymbol{g}(t)$ is proportional to $\boldsymbol{M}\boldsymbol{V}$, that is, to deny the excitation by a control force:

$$\boldsymbol{U}\boldsymbol{g}(t) = \boldsymbol{P}_{\mathrm{e}}(t)\boldsymbol{M}\boldsymbol{V}, \tag{3.8}$$

where $\boldsymbol{P}_{\mathrm{e}}(t) \in \mathbb{R}^{n\times n}$.

Then, the vibration energy is constantly decreasing. The effect of a control force with a constant gain is expressed by a participation factor change, as in Section 3.5. Using a pseudo inverse $\boldsymbol{U}^{-}$, Eq. (3.8) can be expressed as:

$$\boldsymbol{g}(t) = \boldsymbol{U}^{-}\boldsymbol{P}_{\mathrm{e}}(t)\boldsymbol{M}\boldsymbol{V}. \tag{3.9}$$

For example:

$$\boldsymbol{U}_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \qquad \boldsymbol{U}_1^{-} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}. \tag{3.10}$$

And then, if

$$\boldsymbol{P}_{\mathrm{e}}(t)\boldsymbol{M}\boldsymbol{V} = p_{\mathrm{e}}(t)\begin{Bmatrix} m_1 \\ m_2 \\ \cdot \\ m_n \end{Bmatrix}, \qquad \boldsymbol{g}(t) = \boldsymbol{U}_1^{-}\boldsymbol{P}_{\mathrm{e}}(t)\boldsymbol{M}\boldsymbol{V} = p_{\mathrm{e}}(t)\begin{Bmatrix} m_1 \\ m_1 + m_2 \\ \cdot \\ \sum_i m_i \end{Bmatrix}.$$

Thus, if we uniformly provide action–reaction control systems for the reaction to excitation's acceleration at each story of a building, the control force at the bottom story should correspond to the total mass of the building.

Furthermore, if more information for the excitation is provided, we can find other ways, as described in Section 5.

3.5. Constant-gain feedback laws

If all the gains are independent of time, that is, $\boldsymbol{G}_{\mathrm{d}}(t) = \boldsymbol{G}_{\mathrm{d}}$, $\boldsymbol{G}_{\mathrm{v}}(t) = \boldsymbol{G}_{\mathrm{v}}$ and $\boldsymbol{g}(t) = \boldsymbol{g}$, we can understand the control force influence as changes of natural frequencies, damping and participation factors, i.e., $\Delta\omega_i$, Δh_i and $\Delta\beta_i$, respectively. Thus, the deformation FB control force could bring out a dissonant effect, the velocity FB control force dissipates energy, and the excitation FB denies the load due to an excitation.

If $\Delta\omega_i$ is small and the mode vectors have only small changes, these changes are approximated as:

$$\Delta\omega_i = \boldsymbol{\varphi}_i^{\mathrm{T}}\boldsymbol{M}^{-1}\boldsymbol{U}\boldsymbol{G}_{\mathrm{d}}\boldsymbol{\varphi}_i/2\omega_i, \qquad \Delta h_i = \boldsymbol{\varphi}_i^{\mathrm{T}}\boldsymbol{M}^{-1}\boldsymbol{U}\boldsymbol{G}_{\mathrm{v}}\boldsymbol{\varphi}_i/2\omega_i, \qquad \Delta\beta_i = \boldsymbol{\varphi}_i^{\mathrm{T}}\boldsymbol{U}\boldsymbol{g}. \tag{3.11}$$

Compared to the ith natural circular frequency ω_i, damping factor h_i and participation factor β_i of the original model, those of the controlled model ω_i^*, h_i^* and β_i^* become

$$\omega_i^* = \omega_i + \Delta\omega_i, \qquad h_i^* = h_i + \Delta h_i, \qquad \beta_i^* = \beta_i + \Delta\beta_i. \tag{3.12}$$

When the velocity response spectrum of a seismic excitation at ω_i^* with h_i^* is given by $S_V(\omega_i^*, h_i^*)$, the approximated maximum responses of the ith mode $y_i^{*\prime\prime}$, $y_i^{*\prime}$ and y_i^* are estimated by:

$$y_i^{\prime\prime*} = \beta_i^*\omega^* S_V(\omega_i^*, h_i^*), \qquad y_i^{\prime*} = \beta_i^* S_V(\omega_i^*, h_i^*), \qquad y_i^* = \beta_i^* S_V(\omega_i^*, h_i^*)/\omega_i^*. \tag{3.13}$$

Thus, using the least squares method, the maximum responses at the rth DOF of the model, $y_r^{\prime\prime}$, $y_r^{\prime}$ and y_r are approximated by:

$$y_r^{\prime\prime} = \left[\sum_i (\varphi_{ir} y_i^{\prime\prime*})^2\right]^{1/2}, \qquad y_r^{\prime} = \left[\sum_i (\varphi_{ir} y_i^{\prime*})^2\right]^{1/2}, \qquad y_r = \left[\sum_i (\varphi_{ir} y_i^{*})^2\right]^{1/2}, \tag{3.14}$$

where φ_{ir} is the rth component of the ith mode.

While, the maximum value of the jth control force is approximated by:

$$u_j = \left[\sum_i \{(G_{\mathrm{v}j}\varphi_i y_i^{\prime*})^2 + (G_{\mathrm{d}j}\varphi_i y_i^*)^2\} + (g_j w_{\max})^2\right]^{1/2},$$

where $G_{\mathrm{v}j}$, $G_{\mathrm{d}j}$ and g_j are the gains for jth control force, i.e., $\boldsymbol{G}_{\mathrm{v}} = [..G_{\mathrm{v}j}..]$, $\boldsymbol{G}_{\mathrm{d}} = [..G_{\mathrm{d}j}..]$ and $\boldsymbol{g} = [..g_j..]$, and $w_{\max}$ is the maximum of $w(t)$.

This estimate is useful for determining the scales of control systems even if it follows nonlinear control laws.

4. Optimal linear control

4.1. Least quadratic regulator (LQRE) considering excitation influence

On condition that Eq. (2.5) holds, and $\boldsymbol{x}(t_0)$ and $\{w(t)|t \in [t_0, t_1]\}$ are given, let us consider a cost function as:

$$J = \boldsymbol{x}(t_1)^{\mathrm{T}}\boldsymbol{P}_1\boldsymbol{x}(t_1) + \int_{t_0}^{t_1} (\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(t) + \boldsymbol{u}(t)^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{u}(t))\,\mathrm{d}t. \tag{4.1}$$

That is, J is a weighted quadratic norm of structural velocities and displacements and the control force. As a special case, $\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(t)$ can represent vibration energy at t, by assuming:

$$\boldsymbol{P} = \frac{1}{2}\begin{bmatrix} \boldsymbol{M} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{K} \end{bmatrix}. \tag{4.2}$$

As shown in Appendix A, the optimal control force, called the LQRE, is expressed as the sum of the FB and FF terms:

$$\boldsymbol{u}(t) = -\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}(\boldsymbol{S}(t)\boldsymbol{x}(t) + \boldsymbol{f}(t)), \tag{4.3}$$

where $\boldsymbol{S}(t)$ and $\boldsymbol{f}(t)$ are given by:

$$\boldsymbol{S}'(t) + \boldsymbol{S}^{\mathrm{T}}(t)\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{S}(t) - \boldsymbol{S}^{\mathrm{T}}(t)\boldsymbol{B}\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{S}(t) + \boldsymbol{P} = 0, \tag{4.4}$$

$$\boldsymbol{f}'(t) + (\boldsymbol{A}^{\mathrm{T}} - \boldsymbol{S}^{\mathrm{T}}(t)\boldsymbol{B}\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}})\boldsymbol{f}(t) + \boldsymbol{S}(t)\boldsymbol{D}w(t) = 0, \tag{4.5}$$

$$\boldsymbol{S}(t_1) = \boldsymbol{P}_1, \quad \boldsymbol{f}(t_1) = 0. \tag{4.6}$$

$\boldsymbol{S}(t)$ can be obtained as:

$$\boldsymbol{S}(t) = (\boldsymbol{\theta}_{21}(t) + \boldsymbol{\theta}_{22}(t)\boldsymbol{P}_1)(\boldsymbol{\theta}_{11}(t) + \boldsymbol{\theta}_{12}(t)\boldsymbol{P}_1)^{-1}, \tag{4.7}$$

where

$$\begin{bmatrix} \boldsymbol{\theta}_{11}(t) & \boldsymbol{\theta}_{12}(t) \\ \boldsymbol{\theta}_{21}(t) & \boldsymbol{\theta}_{22}(t) \end{bmatrix} = \exp\left((t - t_1)\begin{bmatrix} \boldsymbol{A} & -\boldsymbol{B}\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}} \\ -\boldsymbol{P} & -\boldsymbol{A}^{\mathrm{T}} \end{bmatrix}\right).$$

We have the integrated form for Eq. (4.5) on the condition expressed by Eq. (4.6):

$$\begin{aligned} \boldsymbol{f}(t) &= \exp(-\boldsymbol{Y}(t))\int_t^{t_1} \exp(\boldsymbol{Y}(\tau))\boldsymbol{S}(\tau)\boldsymbol{D}w(\tau)\,\mathrm{d}\tau = \int_t^{t_1} \exp(-\boldsymbol{Y}(t) + \boldsymbol{Y}(\tau))\boldsymbol{S}(\tau)\boldsymbol{D}w(\tau)\,\mathrm{d}\tau \\ &= \int_t^{t_1} \left\{\exp\left(-\int_t^{\tau} (\boldsymbol{A}^{\mathrm{T}} - \boldsymbol{S}(\lambda)\boldsymbol{B}\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}})\,\mathrm{d}\lambda\right)\right\}\boldsymbol{S}(\tau)\boldsymbol{D}w(\tau)\,\mathrm{d}\tau, \end{aligned} \tag{4.8}$$

where

$$\boldsymbol{Y}'(t) = \boldsymbol{A}^{\mathrm{T}} - \boldsymbol{S}(t)\boldsymbol{B}\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}, \quad \boldsymbol{Y}(t_1) = 0, \quad \text{i.e., } \boldsymbol{Y}(t) = \int_t^{t_1} (\boldsymbol{A}^{\mathrm{T}} - \boldsymbol{S}(\tau)\boldsymbol{B}\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}})\,\mathrm{d}\tau. \tag{4.9}$$

Eq. (4.8) shows that the FF term is the convolution with the future excitation information. However, the LQRE is impractical because it requires future excitation information in advance.

Assuming a sampling time Δt, the LQRE for the discrete time system is introduced as well. On condition that Eq. (2.9) holds, let us consider a cost function as:

$$J = \sum_{r=k}^{k+L} [\boldsymbol{x}(r+1)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(r+1) + \boldsymbol{u}(r)^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{u}(r)]. \tag{4.10}$$

As introduced by Fukazawa and Kawahara (1988), the optimal control force is expressed by the sum of the FB and FF terms:

$$\boldsymbol{u}(r) = -(\hat{\boldsymbol{B}}^{\mathrm{T}}\hat{\boldsymbol{S}}(r)\hat{\boldsymbol{B}} + \boldsymbol{Q}^{-1})\hat{\boldsymbol{B}}^{\mathrm{T}}(\hat{\boldsymbol{S}}(r)\hat{\boldsymbol{A}}\boldsymbol{x}(r) + \hat{\boldsymbol{f}}(r)), \tag{4.11}$$

where

$$\hat{\boldsymbol{S}}(r) = \boldsymbol{P} + \hat{\boldsymbol{A}}^{\mathrm{T}}\hat{\boldsymbol{S}}(r+1)\hat{\boldsymbol{A}} - \hat{\boldsymbol{A}}^{\mathrm{T}}\hat{\boldsymbol{S}}(r+1)\hat{\boldsymbol{B}}(\hat{\boldsymbol{B}}^{\mathrm{T}}\hat{\boldsymbol{S}}(r+1)\hat{\boldsymbol{B}} + \boldsymbol{Q})^{-1}\hat{\boldsymbol{B}}^{\mathrm{T}}\hat{\boldsymbol{S}}(r+1)\boldsymbol{A}, \tag{4.12}$$

$$\begin{aligned}\hat{\boldsymbol{f}}(r) &= \sum_{l=r}^{k+L}\left\{\prod_{p=r}^{l}\boldsymbol{Y}(p)\right\}\hat{\boldsymbol{S}}(l)\hat{\boldsymbol{D}}w(l) \\ &= \hat{\boldsymbol{S}}(r)\hat{\boldsymbol{D}}w(r) + \boldsymbol{Y}(r)\hat{\boldsymbol{S}}(r+1)\hat{\boldsymbol{D}}w(r+1) + \cdots + \boldsymbol{Y}(r)\boldsymbol{Y}(r+)\cdots\boldsymbol{Y}(k+L)\hat{\boldsymbol{S}}(k+L)\hat{\boldsymbol{D}}w(k+L),\end{aligned} \tag{4.13}$$

$$\boldsymbol{Y}(r) = \hat{\boldsymbol{A}}^{\mathrm{T}} - \hat{\boldsymbol{A}}^{\mathrm{T}}\hat{\boldsymbol{S}}(r+1)\hat{\boldsymbol{B}}(\hat{\boldsymbol{B}}^{\mathrm{T}}\hat{\boldsymbol{S}}(r+1)\hat{\boldsymbol{B}} + \boldsymbol{Q})^{-1}\hat{\boldsymbol{B}}^{\mathrm{T}}, \tag{4.14}$$

$$\hat{\boldsymbol{S}}(k+L) = \boldsymbol{P}. \tag{4.15}$$

The FF term consists of the instantaneous counter-reaction to the excitation at step r and the convolution with the future excitation information. From Eqs. (4.12)–(4.14), the following relation, which corresponds to Eq. (4.5), is introduced:

$$\hat{\boldsymbol{f}}(r) = \hat{\boldsymbol{S}}(r)\hat{\boldsymbol{D}}w(r) + [\hat{\boldsymbol{A}}^{\mathrm{T}} - \hat{\boldsymbol{A}}^{\mathrm{T}}\hat{\boldsymbol{S}}(r+1)\hat{\boldsymbol{B}}(\hat{\boldsymbol{B}}^{\mathrm{T}}\hat{\boldsymbol{S}}(r+1)\hat{\boldsymbol{B}} + \boldsymbol{Q})^{-1}\hat{\boldsymbol{B}}^{\mathrm{T}}]\hat{\boldsymbol{f}}(r+1). \tag{4.16}$$

4.2. Least quadratic regulator (LQR) for free vibration and a white noise excitation

Assuming that the excitation is a white noise, let us consider a cost function J_{e}, which is an expectation of J assumed by Eq. (4.1), as:

$$J_{\mathrm{e}} = E\left[\boldsymbol{x}(t_1)^{\mathrm{T}}\boldsymbol{P}_1\boldsymbol{x}(t_1) + \int_{t_0}^{t_1}(\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(t) + \boldsymbol{u}(t)^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{u}(t))\,\mathrm{d}t\right], \tag{4.17}$$

where $E[\ ]$ indicates the expectation.

Then, as is well known, the optimal control force is provided by

$$\boldsymbol{u}(t) = -\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{S}(t)\boldsymbol{x}(t). \tag{4.18}$$

where $\boldsymbol{S}(t)$ is defined by Eq. (4.4).

For a discrete time system, assuming J_e as:

$$J_e = E\left[\sum_{r=k}^{N}(\boldsymbol{x}(r+1)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(r+1) + \boldsymbol{u}(r)^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{u}(r))\right], \tag{4.19}$$

and the control force is expressed by:

$$\boldsymbol{u}(r) = -(\boldsymbol{Q} + \hat{\boldsymbol{B}}^{\mathrm{T}}\boldsymbol{S}(r)\hat{\boldsymbol{B}})^{-1}\hat{\boldsymbol{B}}^{\mathrm{T}}\boldsymbol{S}(r)\hat{\boldsymbol{A}}\boldsymbol{x}(r), \tag{4.20}$$

where $\boldsymbol{S}(r)$ is defined by Eq. (4.12).

Hence, the induced control forces do not require the excitation information as a result. That is, they are the same as the optimal control forces introduced for free vibrations. In other words, the control forces provided by Eqs. (4.18) and (4.20) neglect excitation influence, so that the control effects become damping to reduce free vibrations. Therefore, we cannot consider excitation influence on assumption that a seismic excitation is of a white noise.

4.3. Least quadratic regulator (LQRS) for short term

Next, let us consider the case of short term optimization, assuming that Δt is small, $t_0 = 0$, $t_1 = \Delta t$ and $w_0 = w(t_0)$ in Eq. (4.1). At that time, the cost function is given by:

$$J = \boldsymbol{x}(\Delta t)^{\mathrm{T}}\boldsymbol{P}_1\boldsymbol{x}(\Delta t) + \int_0^{\Delta t}(\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(t) + \boldsymbol{u}(t)^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{u}(t))\,\mathrm{d}t. \tag{4.21}$$

Then, we have the following approximation:

$$\exp\left(-\Delta t\begin{bmatrix}\boldsymbol{A} & -\boldsymbol{B}\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}\\ -\boldsymbol{P} & -\boldsymbol{A}^{\mathrm{T}}\end{bmatrix}\right) = \begin{bmatrix}\boldsymbol{I} - \Delta t\boldsymbol{A} + \cdots & \Delta t\boldsymbol{B}\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}} + \cdots\\ \Delta t\boldsymbol{P} + \cdots & \boldsymbol{I} + \Delta t\boldsymbol{A}^{\mathrm{T}} + \cdots\end{bmatrix},$$

$$\begin{aligned}\boldsymbol{S}_0 &\approx (\Delta t\boldsymbol{P} + (\boldsymbol{I} + \Delta t\boldsymbol{A}^{\mathrm{T}})\boldsymbol{P}_1)(\boldsymbol{I} - \Delta t\boldsymbol{A} + \Delta t\boldsymbol{B}\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}_1)^{-1}\\ &= (\boldsymbol{P}_1 + \Delta t(\boldsymbol{P} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}_1))(\boldsymbol{I} + \Delta t(-\boldsymbol{A} + \boldsymbol{B}\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}_1))^{-1},\end{aligned}$$

$$\begin{aligned}\boldsymbol{Y}_0 &= \int_0^{\tau}(\boldsymbol{A}^{\mathrm{T}} - \boldsymbol{S}(\lambda)\boldsymbol{B}\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}})\,\mathrm{d}\lambda = \int_0^{\tau}(\boldsymbol{A}^{\mathrm{T}} - (\boldsymbol{S}_0 + \lambda\boldsymbol{S}_0' + \cdots)\boldsymbol{B}\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}})\,\mathrm{d}\lambda\\ &= \tau(\boldsymbol{A}^{\mathrm{T}} - \boldsymbol{S}_0\boldsymbol{B}\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}) - \frac{\tau^2}{2}\boldsymbol{S}_0'\boldsymbol{B}\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}} + \cdots = \tau\boldsymbol{Y}_0' - \frac{\tau^2}{2}\boldsymbol{S}_0'\boldsymbol{B}\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}} + \cdots,\end{aligned}$$

$$\begin{aligned}&\int_0^{\Delta t}\exp(-\tau\boldsymbol{Y}_0')\,\mathrm{d}\tau = \Delta t\boldsymbol{I} - \frac{\Delta t^2}{2!}\boldsymbol{Y}_0' - \cdots,\\ &\int_0^{\Delta t}\tau\exp(-\tau\boldsymbol{Y}_0')\,\mathrm{d}\tau = \frac{\Delta t^2}{2!}\boldsymbol{I} - \frac{\Delta t^3}{3!}\boldsymbol{Y}_0' - \cdots\end{aligned} \tag{4.22}$$

Eq. (4.8) becomes:

$$\begin{aligned}\boldsymbol{f}(0) &= \int_0^{\Delta t}\left\{\exp\left(-\int_0^{\tau}(\boldsymbol{A}^{\mathrm{T}}-\boldsymbol{S}(\lambda)\boldsymbol{B}\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}})\,\mathrm{d}\lambda\right)\right\}\boldsymbol{S}(\tau)\boldsymbol{D}w(\tau)\,\mathrm{d}\tau \\ &\approx \int_0^{\Delta t}\exp(-\tau\boldsymbol{Y}_0')(\boldsymbol{S}_0+\tau\boldsymbol{S}_0')\boldsymbol{D}(w_0+\tau w_0')\,\mathrm{d}\tau \\ &\approx \int_0^{\Delta t}\exp(-\tau\boldsymbol{Y}_0')\{\boldsymbol{S}_0\boldsymbol{D}w_0+\tau(\boldsymbol{S}_0'\boldsymbol{D}w_0+\boldsymbol{S}_0\boldsymbol{D}w_0')\}\,\mathrm{d}\tau \\ &= \left(\int_0^{\Delta t}\exp(-\tau\boldsymbol{Y}_0')\,\mathrm{d}\tau\right)\boldsymbol{S}_0\boldsymbol{D}w_0+\left(\int_0^{\Delta t}\tau\exp(-\tau\boldsymbol{Y}_0')\,\mathrm{d}\tau\right)(\boldsymbol{S}_0'\boldsymbol{D}w_0+\boldsymbol{S}_0\boldsymbol{D}w_0') \\ &= \left(\Delta t\boldsymbol{I}-\frac{\Delta t^2}{2}\boldsymbol{Y}_0'+\cdots\right)\boldsymbol{S}_0\boldsymbol{D}w_0+\left(\frac{\Delta t^2}{2}\boldsymbol{I}+\cdots\right)(\boldsymbol{S}_0'\boldsymbol{D}w_0+\boldsymbol{S}_0\boldsymbol{D}w_0') \\ &\approx \Delta t\boldsymbol{S}_0\boldsymbol{D}w_0-\frac{\Delta t^2}{2}\{(\boldsymbol{A}^{\mathrm{T}}-\boldsymbol{S}_0\boldsymbol{B}\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}})\boldsymbol{S}_0-\boldsymbol{S}_0'\}\boldsymbol{D}w_0+\frac{\Delta t^2}{2}\boldsymbol{S}_0\boldsymbol{D}w_0' \\ &= \Delta t\boldsymbol{S}_0\boldsymbol{D}w_0+\frac{\Delta t^2}{2}(\boldsymbol{S}_0^{\mathrm{T}}\boldsymbol{A}+\boldsymbol{P})\boldsymbol{D}w_0+\frac{\Delta t^2}{2}\boldsymbol{S}_0\boldsymbol{D}w_0'.\end{aligned} \tag{4.23}$$

Hence, let:

$$\boldsymbol{u}(t) = -\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}\{\boldsymbol{S}_0\boldsymbol{x}(t)+\boldsymbol{f}(t)\}, \tag{4.24}$$

$$\boldsymbol{f}(t) = \Delta t\boldsymbol{S}_0\boldsymbol{D}w(t)+\frac{\Delta t^2}{2}(\boldsymbol{S}_0^{\mathrm{T}}\boldsymbol{A}+\boldsymbol{P})\boldsymbol{D}w(t)+\frac{\Delta t^2}{2}\boldsymbol{S}_0\boldsymbol{D}w'(t). \tag{4.25}$$

Thus, the control force comprises a FB term and a FF term. However, for the precise Δt, the FF term involves only an instantaneous counter-reaction term. The combination of the excitation and the structural dynamics, which is expressed by $-(\Delta t^2/2)\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{S}_0^{\mathrm{T}}\boldsymbol{A}\boldsymbol{D}w(t)$, has little influence on the control force with precise Δt^2. With the same order, the term for acceleration rate contributes to the control force. However, because $(\Delta t^2/2)\boldsymbol{S}_0\boldsymbol{D}w'(t) \sim (\Delta t/2)\boldsymbol{S}_0\boldsymbol{D}(w(t+\Delta t)-w(t))$, some influence of the acceleration at the next step can be considered.

Therefore, the LQRS induces mainly an instantaneous counter-reaction term in addition to the FB term, but little convolution influence caused by future excitation information.

4.4. Least input energy control

On condition that Eq. (2.3) holds, and $x(t_0)$ and $\{w(t)|t\in[t_0,t_1]\}$ are given, let us consider a cost function as:

$$J = \int_{t_0}^{t_1}(\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{D}w(t)+\boldsymbol{u}(t)^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{u}(t))\,\mathrm{d}t. \tag{4.26}$$

For example, if

$$\boldsymbol{P} = \begin{bmatrix}\boldsymbol{M} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}, \tag{4.27}$$

then $\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{D}w(t)$ indicates input energy to a structure by an excitation. Thus, let us call the control strategy minimizing Eq. (4.26) the least input energy control (LIE).

As shown in Appendix B, the optimal control force is expressed by:

$$\boldsymbol{u}(t) = -\tfrac{1}{2}\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{f}(t), \tag{4.28}$$

where

$$\boldsymbol{f}(t_1) = 0, \quad \boldsymbol{f}'(t) = -\boldsymbol{PD}w(t) - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{f}(t). \tag{4.29}$$

Or,

$$\boldsymbol{f}(t) = \int_t^{t_1} \boldsymbol{\Xi}(t-\tau)^{\mathrm{T}}\boldsymbol{PD}w(t)\,\mathrm{d}\tau, \tag{4.30}$$

where $\boldsymbol{\Xi}(t) = \exp(t\boldsymbol{A})$ is defined by Eq. (2.6).

The control force comprises only the convolution term of the excitation. Furthermore, Eqs. (4.29) and (4.30) resemble Eqs. (4.5) and (4.8), respectively, so that similar effects to those induced by those equations are anticipated. In other words, we understand that each term of the control forces by LQRE works to dissipate vibration energy and reduce input energy.

For a discrete time system assuming that Eq. (2.4) holds, let us consider a cost function as:

$$J = \sum_{r=k}^{k+L} (\boldsymbol{x}(r)^{\mathrm{T}}\boldsymbol{P}\hat{\boldsymbol{D}}w(r+1) + \boldsymbol{u}(r)^{\mathrm{T}}\boldsymbol{Qu}(r)). \tag{4.31}$$

Then, the optimal control force is expressed by:

$$\boldsymbol{u}(r) = -\tfrac{1}{2}\boldsymbol{Q}^{-1}\hat{\boldsymbol{B}}^{\mathrm{T}}\hat{\boldsymbol{f}}(r), \tag{4.32}$$

where

$$\hat{\boldsymbol{f}}(r) = \sum_{p=r}^{k+N} (\hat{\boldsymbol{A}}^{\mathrm{T}})^{p-r}\boldsymbol{P}\hat{\boldsymbol{D}}w(p) = \boldsymbol{P}\hat{\boldsymbol{D}}w(r) + \hat{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{P}\hat{\boldsymbol{D}}w(r+1) + \cdots + (\hat{\boldsymbol{A}}^{\mathrm{T}})^{k+L-1}\boldsymbol{P}\hat{\boldsymbol{D}}w(k+L). \tag{4.33}$$

Therefore, the LIE induces only a term corresponding to future excitation information.

As in the foregoing, the LQRE, which aims at reducing vibration energy, induces the FB term and the convolution term with future excitation information. This convolution term cannot be induced by the LQR, i.e., on assumption that a seismic excitation is of a white noise. The LQRS induces an instantaneous counter-reaction term as well as a FB term, but only a small term corresponding to future excitation information, while the LIE induces only the convolution term with future excitation information. That is, to incorporate excitation influence into a control strategy, we must acknowledge future excitation information or something to substitute for it. Thus, Section 5 examines control strategies, assuming dynamic characteristics of a seismic excitation by a state equation model.

5. Control strategy for excitation model

5.1. Excitation model

Let us assume that excitation information is modeled as a q-dimensional state equation model:

$$\boldsymbol{v}'(t) = \boldsymbol{Hv}(t) + \boldsymbol{L}\zeta(t), \tag{5.1}$$

$$w(t) = \boldsymbol{F}^{\mathrm{T}}\boldsymbol{v}(t) + \zeta(t), \tag{5.2}$$

where $\boldsymbol{v}(t) \in \mathbb{R}^q$, $\boldsymbol{H} \in \mathbb{R}^{q\times q}$, $\boldsymbol{F} \in \mathbb{R}^q$ and $\zeta(t) \in \mathbb{R}$; the real parts of the eigenvalues of $\boldsymbol{H}$ should be less than 0; and $\zeta(t)$ represents an error.

Or, assuming sampling time Δt, let excitation information be modeled by a discrete q-dimensional state equation model:

$$\boldsymbol{v}(r+1) = \hat{\boldsymbol{H}}\boldsymbol{v}(r) + \hat{\boldsymbol{L}}\zeta(r), \tag{5.3}$$

$$w(r) = \hat{\boldsymbol{F}}^{\mathrm{T}}\boldsymbol{v}(r) + \zeta(r), \tag{5.4}$$

where $\boldsymbol{v}(r) \in \mathbb{R}^q$, $\hat{\boldsymbol{H}} \in \mathbb{R}^{q\times q}$, $\hat{\boldsymbol{F}} \in \mathbb{R}^q$ and $\zeta(r) \in \mathbb{R}$; the absolute values of eigenvalues of $\hat{\boldsymbol{H}}$ should be less than 1.0; and $\zeta(r)$ represents an error.

To obtain coefficients for the model, see Yamada (1999b).

5.2. Extended state feedback control

Using the foregoing excitation model, we can develop an extended FB control law for a structural model expressed by Eq. (2.5). That is, let us assume:

$$\boldsymbol{u}(t) = \boldsymbol{G}\boldsymbol{x}(t) + \boldsymbol{Y}\boldsymbol{v}(t),$$

where $\boldsymbol{G} \in \mathbb{R}^{m\times n}$ and $\boldsymbol{Y} \in \mathbb{R}^{m\times q}$ indicate FB gains.

Then, the structural dynamics is expressed by:

$$\boldsymbol{x}'(t) = (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{G})\boldsymbol{x}(t) + (\boldsymbol{D}\boldsymbol{F}^{\mathrm{T}} + \boldsymbol{B}\boldsymbol{Y})\boldsymbol{v}(t) + \boldsymbol{D}\zeta(t). \tag{5.5}$$

Using a Laplace transform:

$$s\tilde{\boldsymbol{x}}(s) - \boldsymbol{x}_0 = (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{G})\tilde{\boldsymbol{x}}(s) + (\boldsymbol{D}\boldsymbol{F}^{\mathrm{T}} + \boldsymbol{B}\boldsymbol{Y})\tilde{\boldsymbol{v}}(s) + \boldsymbol{D}\tilde{\zeta}(s), \tag{5.6}$$

$$s\tilde{\boldsymbol{v}}(s) - \boldsymbol{v}_0 = \boldsymbol{H}\tilde{\boldsymbol{v}}(s) + \tilde{\zeta}(s), \tag{5.7}$$

where $\tilde{\boldsymbol{x}}(s)$, $\tilde{\boldsymbol{v}}(s)$ and $\tilde{\zeta}(s)$ indicate the Laplace transform of $\boldsymbol{x}(s)$, $\boldsymbol{v}(s)$ and $\zeta(s)$, respectively.

Hence:

$$\begin{aligned}\tilde{\boldsymbol{x}}(s) = {} & \{s\boldsymbol{I} - (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{G})\}^{-1}\boldsymbol{x}_0 + \{s\boldsymbol{I} - (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{G})\}^{-1}(\boldsymbol{D}\boldsymbol{F}^{\mathrm{T}} + \boldsymbol{B}\boldsymbol{Y})(s\boldsymbol{I} - \boldsymbol{H})^{-1}\boldsymbol{v}_0 \\ & + \{s\boldsymbol{I} - (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{G})\}^{-1}\{(\boldsymbol{D}\boldsymbol{F}^{\mathrm{T}} + \boldsymbol{B}\boldsymbol{Y})(s\boldsymbol{I} - \boldsymbol{H})^{-1}\boldsymbol{L} + \boldsymbol{D}\}\tilde{\zeta}(s).\end{aligned} \tag{5.8}$$

The first term in the right-hand side indicates the response to the initial state, the second term represents the convolution with excitation information, and the last term represents the influence of the error.

Letting $\boldsymbol{\Pi} = \mathrm{diag}\{\varpi_i\} = \boldsymbol{\Phi}^{\mathrm{T}}(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{G})\boldsymbol{\Phi}$, $\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\Phi} = \boldsymbol{I}$, $\boldsymbol{\Phi} = \{..\boldsymbol{\phi}_i..\}$, $\Lambda = \mathrm{diag}\{\lambda_k\} = \boldsymbol{\Psi}^{\mathrm{T}}\boldsymbol{H}\Psi$, $\boldsymbol{\Psi} = \{..\boldsymbol{\psi}_k..\}$, $\boldsymbol{\Psi}^{\mathrm{T}}\boldsymbol{\Psi} = \boldsymbol{I}$, Eq. (5.8) is decomposed to:

$$\begin{aligned}\boldsymbol{\Phi}^{\mathrm{T}}\tilde{\boldsymbol{x}}(s) = {} & (s\boldsymbol{I} - \boldsymbol{\Pi})^{-1}\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{x}_0 + (s\boldsymbol{I} - \boldsymbol{\Pi})^{-1}\boldsymbol{\Phi}^{\mathrm{T}}(\boldsymbol{D}\boldsymbol{F}^{\mathrm{T}} + \boldsymbol{B}\boldsymbol{Y})\boldsymbol{\Psi}(s\boldsymbol{I} - \Lambda)^{-1}\boldsymbol{\Psi}^{\mathrm{T}}\boldsymbol{v}_0 + (s\boldsymbol{I} - \boldsymbol{\Pi})^{-1} \\ & \times \{\boldsymbol{\Phi}^{\mathrm{T}}(\boldsymbol{D}\boldsymbol{F}^{\mathrm{T}} + \boldsymbol{B}\boldsymbol{Y})\boldsymbol{\Psi}(s\boldsymbol{I} - \Lambda)^{-1}\boldsymbol{\Psi}^{\mathrm{T}}\boldsymbol{L} + \boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{D}\}\tilde{\zeta}(s).\end{aligned} \tag{5.9}$$

Let us define the participation factors β_{ik}, the response sources $\chi_{ik}(t)$ of the ith state mode to the kth excitation mode, and its Laplace transform $\tilde{\chi}_{ik}(s)$ as:

$$\begin{aligned}\beta_{ik} &= \boldsymbol{\phi}_i^{\mathrm{T}}(\boldsymbol{D}\boldsymbol{F}^{\mathrm{T}} + \boldsymbol{B}\boldsymbol{Y})\boldsymbol{\psi}_k, \\ \tilde{\chi}_{ik}(s) &= (s - \varpi_i)^{-1}(s - \lambda_k)^{-1} = \left\{(s - \varpi_i)^{-1} - (s - \lambda_k)^{-1}\right\}\Big/(\varpi_i - \lambda_k), \\ \chi_{ik}(t) &= \left(\frac{1}{\varpi_i}\exp(\varpi_i t) - \frac{1}{\lambda_k}\exp(\lambda_k t)\right)\Big/(\varpi_i - \lambda_k).\end{aligned} \tag{5.10}$$

Then:

$$x_i^*(t) = \boldsymbol{\phi}_i^{\mathrm{T}}\boldsymbol{x}(t) = x_{i0}^*\exp(\varpi_i t) + \sum_k \beta_{ik}\chi_{ik}(t)v_{0k}^* + \zeta_i(t), \tag{5.11}$$

where $\boldsymbol{\phi}_i^{\mathrm{T}} \boldsymbol{x}_0 = x_{0k}^*$, $\boldsymbol{\psi}_k^{\mathrm{T}} \boldsymbol{v}_0 = v_{0k}^*$, and $\zeta_i(t)$ is an error to the ith mode.

If the real part of ϖ_i is negative and small enough, the responses are rapidly reduced. When β_{ik} is small, the influence of the response source becomes small. The case where each component of $(\boldsymbol{DF}^{\mathrm{T}} + \boldsymbol{BY})$ is small, i.e., a control force works to deny an excitation source, is one such case. Furthermore, a response source is small when ϖ_i is far from λ_i. The structural responses then become small. This case is called dissonance.

We can view the foregoing results in the time domain as well. Let us express the structural dynamics and the excitation model in an augmented space as:

$$\begin{Bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{v}(t) \end{Bmatrix}' = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{DF}^{\mathrm{T}} \\ \boldsymbol{0} & \boldsymbol{H} \end{bmatrix} \begin{Bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{v}(t) \end{Bmatrix} + \begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{u}(t) + \begin{bmatrix} \boldsymbol{D} \\ \boldsymbol{L} \end{bmatrix} \zeta(t). \tag{5.12}$$

When the extended FB control law is assumed, we have

$$\begin{Bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{v}(t) \end{Bmatrix} = \begin{bmatrix} \boldsymbol{A} + \boldsymbol{BG} & \boldsymbol{DF}^{\mathrm{T}} + \boldsymbol{BY} \\ \boldsymbol{0} & \boldsymbol{H} \end{bmatrix} \begin{Bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{v}(t) \end{Bmatrix} + \begin{bmatrix} \boldsymbol{D} \\ \boldsymbol{L} \end{bmatrix} \zeta(t). \tag{5.13}$$

Then, the solution is given by:

$$\begin{Bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{v}(t) \end{Bmatrix} = \exp(t\tilde{\boldsymbol{A}}) \begin{Bmatrix} \boldsymbol{x}(0) \\ \boldsymbol{v}(0) \end{Bmatrix} + \exp(t\tilde{\boldsymbol{A}}) * \begin{bmatrix} \boldsymbol{D} \\ \boldsymbol{L} \end{bmatrix} \zeta(t), \tag{5.14}$$

where $*$ indicates convolution and:

$$\exp(t\tilde{\boldsymbol{A}}) = \begin{bmatrix} \exp(t(\boldsymbol{A} + \boldsymbol{BG})) & \sum_{n=1}^{\infty} \frac{t^n}{n!} \left(\sum_{r=0}^{n-1} (\boldsymbol{A} + \boldsymbol{BG})^{n-1-r} (\boldsymbol{DF}^{\mathrm{T}} + \boldsymbol{BY}) \boldsymbol{H}^r \right) \\ 0 & \exp(t\boldsymbol{H}) \end{bmatrix}.$$

Thus, as considered in the frequency domain, if the real parts of the eigenvalues of $(\boldsymbol{A} + \boldsymbol{BG})$ are negative and small enough, the responses are quickly reduced. If the components of $(\boldsymbol{DF}^{\mathrm{T}} + \boldsymbol{BY})$ are small, or if the eigenvalues of $(\boldsymbol{A} + \boldsymbol{BG})$ are far from those of $\boldsymbol{H}$, the responses have less excitation influence. In other words, we have three strategies: (i) to dampen the responses to the initial conditions at each moment, (ii) to activate a control force neglecting the excitation to a structure, and (iii) to isolate the structural natural frequency from the frequency of the excitation sources.

5.3. Extended least quadratic regulator (ELQR)

We can extend the LQR for the excitation information expressed by a state equation. Let us call it the ELQR. Firstly, as expressed by Eq. (5.12), let the structural and excitation dynamics be expressed by a state equation in an augmented space:

$$\boldsymbol{z}'(t) = \tilde{\boldsymbol{A}} \boldsymbol{z}(t) + \tilde{\boldsymbol{B}} \boldsymbol{u}(t) + \tilde{\boldsymbol{D}} \zeta(t), \tag{5.15}$$

where

$$\boldsymbol{z}(t) = \begin{Bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{v}(t) \end{Bmatrix}, \qquad \tilde{\boldsymbol{A}} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{DF}^{\mathrm{T}} \\ \boldsymbol{0} & \boldsymbol{H} \end{bmatrix}, \qquad \tilde{\boldsymbol{B}} = \begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{0} \end{bmatrix}, \qquad \tilde{\boldsymbol{D}} = \begin{bmatrix} \boldsymbol{D} \\ \boldsymbol{L} \end{bmatrix},$$

i.e., $\boldsymbol{z}(t) \in \mathbb{R}^{2n+q}$, $\tilde{\boldsymbol{A}} \in \mathbb{R}^{(2n+q)\times(2n+q)}$, $\tilde{\boldsymbol{B}} \in \mathbb{R}^{(2n+q)\times m}$ and $\tilde{\boldsymbol{D}} \in \mathbb{R}^{2n+q}$.

Next, assume a cost function as:

$$J_e = E\left[\boldsymbol{x}(t_1)^{\mathrm{T}}\boldsymbol{P}_1\boldsymbol{x}(t_1) + \int_{t_0}^{t_1}(\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(t) + \boldsymbol{u}(t)^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{u}(t))\,\mathrm{d}t\right]$$
$$= E\left[\boldsymbol{z}(t_1)^{\mathrm{T}}\tilde{\boldsymbol{P}}_1\boldsymbol{z}(t_1) + \int_{t_0}^{t_1}(\boldsymbol{z}(t)^{\mathrm{T}}\tilde{\boldsymbol{P}}\boldsymbol{z}(t) + \boldsymbol{u}(t)^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{u}(t))\,\mathrm{d}t\right], \quad (5.16)$$

where

$$\tilde{\boldsymbol{P}} = \begin{bmatrix} \boldsymbol{P} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}, \qquad \tilde{\boldsymbol{P}}_1 = \begin{bmatrix} \boldsymbol{P}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix},$$

i.e., $\tilde{\boldsymbol{P}}$ and $\tilde{\boldsymbol{P}}_1 \in \mathbb{R}^{(2n+q)\times(2n+q)}$.

Then, the optimal control force minimizing the cost function is expressed by:

$$\boldsymbol{u}(t) = -\boldsymbol{Q}^{-1}\tilde{\boldsymbol{B}}^{\mathrm{T}}\tilde{\boldsymbol{S}}(t)\boldsymbol{z}(t) = -\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}(\tilde{\boldsymbol{S}}_{11}\boldsymbol{x}(t) + \tilde{\boldsymbol{S}}_{12}\boldsymbol{v}(t)), \quad (5.17)$$

where

$$\tilde{\boldsymbol{S}}(t) = \begin{bmatrix} \tilde{\boldsymbol{S}}_{11} & \tilde{\boldsymbol{S}}_{12} \\ \tilde{\boldsymbol{S}}_{21} & \tilde{\boldsymbol{S}}_{22} \end{bmatrix}, \; \tilde{\boldsymbol{S}} \in \mathbb{R}^{(2n+q)\times(2n+q)}$$

is given by:

$$\tilde{\boldsymbol{S}}'(t) + \tilde{\boldsymbol{S}}^{\mathrm{T}}(t)\tilde{\boldsymbol{A}} + \tilde{\boldsymbol{A}}^{\mathrm{T}}\tilde{\boldsymbol{S}}(t) - \tilde{\boldsymbol{S}}^{\mathrm{T}}(t)\tilde{\boldsymbol{B}}\boldsymbol{Q}^{-1}\tilde{\boldsymbol{B}}^{\mathrm{T}}\tilde{\boldsymbol{S}}(t) + \tilde{\boldsymbol{P}} = 0, \quad (5.18)$$

$$\tilde{\boldsymbol{S}}(t) = \tilde{\boldsymbol{P}}_1. \quad (5.19)$$

In fact, $\tilde{\boldsymbol{S}}_{11}(t) = \boldsymbol{S}(t)$, so that the ELQR comprises the FF term $-\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}\tilde{\boldsymbol{S}}_{12}\boldsymbol{v}(t)$ in addition to the FB term by the LQR.

We can introduce the ELQR for the discrete time system (Yamada and Kobori, 1996). That is, let us express a state equation in an augmented space as:

$$\boldsymbol{z}(r+1) = \breve{\boldsymbol{A}}\boldsymbol{z}(r) + \breve{\boldsymbol{B}}\boldsymbol{u}(r) + \breve{\boldsymbol{D}}\zeta(r), \quad (5.20)$$

where

$$\boldsymbol{z}(r) = \begin{Bmatrix} \boldsymbol{x}(r) \\ \boldsymbol{v}(r) \end{Bmatrix}, \qquad \breve{\boldsymbol{A}} = \begin{bmatrix} \hat{\boldsymbol{A}} & \hat{\boldsymbol{D}}\hat{\boldsymbol{F}}^{\mathrm{T}} \\ \boldsymbol{0} & \hat{\boldsymbol{H}} \end{bmatrix}, \qquad \breve{\boldsymbol{B}} = \begin{bmatrix} \hat{\boldsymbol{B}} \\ \boldsymbol{0} \end{bmatrix}, \qquad \breve{\boldsymbol{D}} = \begin{bmatrix} \hat{\boldsymbol{D}} \\ \hat{\boldsymbol{L}} \end{bmatrix}.$$

Let the cost function be:

$$J_e = E\left[\sum_{r=k}^{N}(\boldsymbol{x}(r+1)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(r+1) + \boldsymbol{u}(r)^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{u}(r))\right]$$
$$= E\left[\sum_{r=k}^{N}(\boldsymbol{z}(r+1)^{\mathrm{T}}\breve{\boldsymbol{P}}\boldsymbol{z}(r+1) + \boldsymbol{u}(r)^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{u}(r))\right], \quad (5.21)$$

where

$$\breve{\boldsymbol{P}} = \begin{bmatrix} \boldsymbol{P} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}.$$

Then, the control force is expressed by:

$$\boldsymbol{u}(r) = -(\boldsymbol{Q} + \breve{\boldsymbol{B}}^{\mathrm{T}}\breve{\boldsymbol{S}}(r)\breve{\boldsymbol{B}})^{-1}\breve{\boldsymbol{B}}^{\mathrm{T}}\breve{\boldsymbol{S}}(r)\breve{\boldsymbol{A}}\boldsymbol{z}(r), \quad (5.22)$$

where $\breve{\boldsymbol{S}}(t)$ is defined by:

$$\breve{\boldsymbol{S}}(r) = \breve{\boldsymbol{P}} + \breve{\boldsymbol{A}}^{\mathrm{T}}\breve{\boldsymbol{S}}(r+1)\breve{\boldsymbol{A}} - \breve{\boldsymbol{A}}^{\mathrm{T}}\breve{\boldsymbol{S}}(r+1)\breve{\boldsymbol{B}}(\breve{\boldsymbol{B}}^{\mathrm{T}}\breve{\boldsymbol{S}}(r+1)\breve{\boldsymbol{B}} + \boldsymbol{Q})^{-1}\breve{\boldsymbol{B}}^{\mathrm{T}}\breve{\boldsymbol{S}}(r+1)\breve{\boldsymbol{A}}. \tag{5.23}$$

In fact, Eq. (5.22) can be decomposed to:

$$\boldsymbol{u}(r) = -(\boldsymbol{Q} + \hat{\boldsymbol{B}}^{\mathrm{T}}\breve{\boldsymbol{S}}_{11}(r)\hat{\boldsymbol{B}})^{-1}\hat{\boldsymbol{B}}^{\mathrm{T}}(\breve{\boldsymbol{S}}_{11}(r)\hat{\boldsymbol{A}}\boldsymbol{x}(r) + \breve{\boldsymbol{S}}_{11}(r)\hat{\boldsymbol{D}}\hat{\boldsymbol{F}}^{\mathrm{T}}\boldsymbol{v}(r) + \breve{\boldsymbol{S}}_{12}(r)\hat{\boldsymbol{H}}\boldsymbol{v}(r)), \tag{5.24}$$

where

$$\breve{\boldsymbol{S}} = \begin{bmatrix} \breve{\boldsymbol{S}}_{11} & \breve{\boldsymbol{S}}_{12} \\ \breve{\boldsymbol{S}}_{21} & \breve{\boldsymbol{S}}_{22} \end{bmatrix}.$$

Thus, we can write Eq. (5.24) as:

$$\boldsymbol{u}(r) = \boldsymbol{G}^{\mathrm{FB}}\boldsymbol{x}(r) + \boldsymbol{G}^{\mathrm{IC}}\boldsymbol{v}(r) + \boldsymbol{G}^{\mathrm{FF}}\boldsymbol{v}(r) = \boldsymbol{u}^{\mathrm{FB}}(r) + \boldsymbol{u}^{\mathrm{IC}}(r) + \boldsymbol{u}^{\mathrm{FF}}(r), \tag{5.25}$$

where

$$\begin{aligned} \boldsymbol{G}^{\mathrm{FB}} &= -(\boldsymbol{Q} + \hat{\boldsymbol{B}}^{\mathrm{T}}\breve{\boldsymbol{S}}_{11}(r)\hat{\boldsymbol{B}})^{-1}\hat{\boldsymbol{B}}^{\mathrm{T}}\breve{\boldsymbol{S}}_{11}(r)\hat{\boldsymbol{A}}, & \boldsymbol{u}^{\mathrm{FB}}(k) &= \boldsymbol{G}^{\mathrm{FB}}\boldsymbol{x}(k), \\ \boldsymbol{G}^{\mathrm{IC}} &= -(\boldsymbol{Q} + \hat{\boldsymbol{B}}^{\mathrm{T}}\breve{\boldsymbol{S}}_{11}(r)\hat{\boldsymbol{B}})^{-1}\hat{\boldsymbol{B}}^{\mathrm{T}}\breve{\boldsymbol{S}}_{11}(r)\hat{\boldsymbol{D}}\hat{\boldsymbol{F}}^{\mathrm{T}}, & \boldsymbol{u}^{\mathrm{IC}}(k) &= \boldsymbol{G}^{\mathrm{IC}}\boldsymbol{v}(k), \\ \boldsymbol{G}^{\mathrm{FF}} &= -(\boldsymbol{Q} + \hat{\boldsymbol{B}}^{\mathrm{T}}\breve{\boldsymbol{S}}_{11}(r)\hat{\boldsymbol{B}})^{-1}\hat{\boldsymbol{B}}^{\mathrm{T}}\breve{\boldsymbol{S}}_{12}(r)\hat{\boldsymbol{H}}, & \boldsymbol{u}^{\mathrm{FF}}(k) &= \boldsymbol{G}^{\mathrm{FF}}\boldsymbol{v}(k). \end{aligned}$$

It should be noted that $\breve{\boldsymbol{S}}_{11}(r) = \hat{\boldsymbol{S}}(r)$, i.e., equal to the solution of the Ricatti equation for the LQR. Thus, the first term works as damping transient responses. Because $\hat{\boldsymbol{F}}^{\mathrm{T}}\boldsymbol{v}(k) \approx w(k)$, the second term works to deny excitation. The third term corresponds to excitation dynamics, which prepares for future excitation influence. Thus, the three terms in Eq. (5.25) are called the FB, instantaneous counter action and FF terms, respectively.

6. Optimal control under constraints

This section describes optimal control strategies induced under more restricted conditions. First, control force amplitude is explicitly restrained. Second, a higher-order norm for the control forces is considered. Next, the optimization for variable elements is taken account of. Furthermore, control for the VDEs with ASEs is examined.

6.1. Optimization for restrained control force amplitude

Assuming structural dynamics for $t \in [t_0, t_1]$ expressed by Eq. (2.5), with an initial condition:

$$\boldsymbol{x}(t_0) = \boldsymbol{x}_0. \tag{6.1}$$

Let us obtain the optimal control force that minimizes the cost function J:

$$J = F_1(\boldsymbol{x}(t_1), t_1) + \int_{t_0}^{t_1} F(\boldsymbol{x}(t), \boldsymbol{u}(t), t)\,\mathrm{d}t, \tag{6.2}$$

under the constraint

$$|u_j(t)| \leqslant \upsilon, \tag{6.3}$$

where $\boldsymbol{u}(t) = \{u_j(t)\}$ and t_1 is fixed.

Based on the optimization theory, Hamiltonian is defined as:

$$H = F(\boldsymbol{x}(t), \boldsymbol{u}(t), t) + \boldsymbol{\lambda}(t)^{\mathrm{T}}(\boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{D}w(t)), \tag{6.4}$$

where $\boldsymbol{\lambda}(t) \in \mathbb{R}^{2n}$.

The boundary condition for $\boldsymbol{\lambda}(t)$ is given by:

$$\boldsymbol{\lambda}(t_1) = \partial F_1(\boldsymbol{x}(t_1), t_1)/\partial \boldsymbol{x}. \tag{6.5}$$

Then, let us obtain the optimal control forces for the following two cases:

Case A

Assuming $F(\boldsymbol{x}(t), \boldsymbol{u}(t), t) = F_A(\boldsymbol{x}(t))$,

$$\partial H/\partial \boldsymbol{u} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{\lambda}(t). \tag{6.6}$$

Thus, $H_{\min}$ is given by:

$$\text{(i) } u_j = \upsilon \quad \text{for } \boldsymbol{B}_j^{\mathrm{T}}\boldsymbol{\lambda}(t) < 0; \ \text{(ii) } u_j = -\upsilon \quad \text{for } \boldsymbol{B}_j^{\mathrm{T}}\boldsymbol{\lambda}(t) > 0; \quad \text{i.e., } u_j = -\upsilon\,\mathrm{sgn}(\boldsymbol{B}_j^{\mathrm{T}}\boldsymbol{\lambda}(t)). \tag{6.7}$$

Then, the Euler equations are:

$$\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\{.. - \upsilon\,\mathrm{sgn}(\boldsymbol{B}_j^{\mathrm{T}}\boldsymbol{\lambda}(t))..\}^{\mathrm{T}} + \boldsymbol{D}w(t), \tag{6.8}$$

$$\boldsymbol{\lambda}'(t) = -\partial H/\partial \boldsymbol{x} = -\partial F_A(\boldsymbol{x}(t))/\partial \boldsymbol{x} - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\lambda}(t). \tag{6.9}$$

The optimal solution $\boldsymbol{x}(t)$ is provided by solving Eqs. (6.8) and (6.9) under two-point boundary conditions given by Eqs. (6.1) and (6.5) and excitation influence, which is in fact difficult because a sign function is included. Furthermore, the control force expressed by a sign function produces residual vibrations, as shown in Section 8.

Case B

Let us assume that $F(\boldsymbol{x}(t), \boldsymbol{u}(t), t) = F_B(\boldsymbol{x}(t)) + (1/2)\boldsymbol{u}(t)^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{u}(t)$ and $\boldsymbol{Q} = \mathrm{diag}\{q_j\}$. Then,

$$\partial H/\partial \boldsymbol{u} = \boldsymbol{Q}\boldsymbol{u}(t) + \boldsymbol{B}^{\mathrm{T}}\boldsymbol{\lambda}(t). \tag{6.10}$$

Letting $\boldsymbol{B}^{\mathrm{T}} = \{..\boldsymbol{B}_j^{\mathrm{T}}..\}$, $H_{\min}$ is given by:

(i) $u_j = \upsilon$ for $q_j^{-1}\boldsymbol{B}_j^{\mathrm{T}}\boldsymbol{\lambda}(t) < -\upsilon$;
(ii) $u_j = -q_j^{-1}\boldsymbol{B}_j^{\mathrm{T}}\boldsymbol{\lambda}(t)$ for $|q_j^{-1}\boldsymbol{B}_j^{\mathrm{T}}\boldsymbol{\lambda}(t)| < \upsilon$;
(iii) $u_j = -\upsilon$ for $q_j^{-1}\boldsymbol{B}_j^{\mathrm{T}}\boldsymbol{\lambda}(t) > \upsilon$,

which are written as:

$$u_j = -\upsilon\,\mathrm{sat}(q_j^{-1}\boldsymbol{B}_j^{\mathrm{T}}\boldsymbol{\lambda}(t)/\upsilon). \tag{6.11}$$

The bound for region (ii) is provided by $|q_j^{-1}\boldsymbol{B}_j^{\mathrm{T}}\boldsymbol{\lambda}(t)| = \upsilon$. To obtain $\boldsymbol{\lambda}(t)$, we must solve the following Euler equations under two-point boundary conditions.

$$\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\{.. - \upsilon\,\mathrm{sat}(q_j^{-1}\boldsymbol{B}_j^{\mathrm{T}}\boldsymbol{\lambda}(t)/\upsilon)..\}^{\mathrm{T}} + \boldsymbol{D}w(t), \tag{6.12}$$

$$\boldsymbol{\lambda}'(t) = -\partial F_B(\boldsymbol{x}(t))/\partial \boldsymbol{x} - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\lambda}(t). \tag{6.13}$$

For the case $F_B(\boldsymbol{x}(t)) = (1/2)\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(t)$, the LQRE provides the optimal control force if all control forces are in region (ii). To consider excitation influence, we may assume a state equation model for excitation information as shown in Section 5. However, if one or more control force reaches its limit, it is difficult to analytically solve Eqs. (6.12) and (6.13) under two-point boundary conditions.

If we substitute tanh( ) for sat( ), the control force is expressed by:

$$u_j(t) = -\upsilon\tanh(\theta q_j^{-1}\boldsymbol{B}_j^{\mathrm{T}}\boldsymbol{\lambda}(t)/\upsilon), \tag{6.14}$$

where θ adjusts the smoothness.

Then,

$$u_j'(t) = -\theta q_j^{-1}\boldsymbol{B}_j^{\mathrm{T}}\boldsymbol{\lambda}'(t)\,\mathrm{sech}(\theta q_j^{-1}\boldsymbol{B}_j^{\mathrm{T}}\boldsymbol{\lambda}(t)/\upsilon) = -\theta q_j^{-1}\boldsymbol{B}_j^{\mathrm{T}}\boldsymbol{\lambda}'(t)\left\{1 - \left(\frac{1}{\upsilon}u_j(t)\right)^2\right\}. \tag{6.15}$$

Compared to the LQRE control force given by differentiating Eq. (A.4), the term $\theta\{1-((1/v)u_j(t))^2\}$ restrains the control force rate, i.e., increment per unit time. Thus, adding this nonlinear term to the linear control law expressed by a differential equation, i.e., an indirect law, could restrain the control force amplitude.

6.2. Optimization for higher-order norms

A nonlinear control law restraining control force amplitude can be also induced for a cost function involving higher-order norms. For example, for the structural dynamics defined by Eq. (2.5), let us assume the cost function defined by:

$$J=\frac{1}{2}\|\boldsymbol{x}(t_1)\|^2_{\boldsymbol{P}_1}+\int_{t_0}^{t_1}\left\{\frac{1}{2}\|\boldsymbol{x}(t)\|^2_{\boldsymbol{P}}+\frac{1}{2}\|\boldsymbol{u}(t)\|^2_{\boldsymbol{Q}}+\frac{1}{4}\|\boldsymbol{u}(t)\|^4_{\boldsymbol{R}}\right\}\mathrm{d}t, \tag{6.16}$$

where $\boldsymbol{P}_1\in\mathbb{R}^{2n\times 2n}$, $\boldsymbol{P}\in\mathbb{R}^{2n\times 2n}$, $\boldsymbol{Q}\in\mathbb{R}^{m\times m}$, $\boldsymbol{R}\in\mathbb{R}^{m\times m}$; and $\|\ \|^r_{\boldsymbol{W}}$ denotes rth-order norm weighted by $\boldsymbol{W}$.

Then, assuming $\boldsymbol{\lambda}(t)\in\mathbb{R}^{2n}$, Hamiltonian is given by:

$$H=\frac{1}{2}\|\boldsymbol{x}(t)\|^2_{\boldsymbol{P}}+\frac{1}{2}\|\boldsymbol{u}(t)\|^2_{\boldsymbol{Q}}+\frac{1}{4}\|\boldsymbol{u}(t)\|^4_{\boldsymbol{R}}+\boldsymbol{\lambda}^{\mathrm{T}}\{\boldsymbol{A}\boldsymbol{x}(t)+\boldsymbol{B}\boldsymbol{u}(t)+\boldsymbol{D}w(t)\}. \tag{6.17}$$

Thus, the optimal values should satisfy:

$$-\boldsymbol{\lambda}'(t)=\partial H/\partial\boldsymbol{x}=\boldsymbol{P}\boldsymbol{x}(t)+\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\lambda}(t), \tag{6.18}$$

$$0=\partial H/\partial\boldsymbol{u}=\boldsymbol{Q}\boldsymbol{u}(t)+\boldsymbol{R}\boldsymbol{u}^3(t)+\boldsymbol{B}^{\mathrm{T}}\boldsymbol{\lambda}(t), \tag{6.19}$$

$$\boldsymbol{\lambda}(t_1)=\boldsymbol{P}_1\boldsymbol{x}(t_1), \tag{6.20}$$

where $\boldsymbol{u}^3(t)=\{u_j(t)^3\}$.

Differentiating Eq. (6.19) provides an indirect control law as:

$$\boldsymbol{u}'(t)=-(\boldsymbol{Q}+3\boldsymbol{R}\,\mathrm{diag}\{u_i(t)^2\})^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{\lambda}'(t), \tag{6.21}$$

where $\mathrm{diag}\{u_i(t)^2\}\in\mathbb{R}^{m\times m}$ denotes a diagonal matrix whose components are square of each control force component.

Comparing the above to Eq. (6.15), $(\boldsymbol{Q}+3\boldsymbol{R}\,\mathrm{diag}\{u_i(t)^2\})^{-1}$ restrains control force amplitude. However, $\boldsymbol{\lambda}(t)$ should be nonlinear because Eq. (6.19) is nonlinear. If $\boldsymbol{\lambda}(t)$ is expressed by:

$$\boldsymbol{\lambda}(t)=\boldsymbol{S}(t)\boldsymbol{x}(t)+\xi(t), \tag{6.22}$$

with the end condition as:

$$\xi(t_1)=0, \tag{6.23}$$

we have:

$$\boldsymbol{\lambda}'(t)=\boldsymbol{S}'(t)\boldsymbol{x}(t)+\boldsymbol{S}(t)\{\boldsymbol{A}\boldsymbol{x}(t)+\boldsymbol{B}\boldsymbol{u}(t)+\boldsymbol{D}w(t)\}+\xi'(t). \tag{6.24}$$

While, Eq. (6.19) can be expressed as:

$$\begin{aligned}\boldsymbol{u}(t)&=-(\boldsymbol{Q}+\boldsymbol{R}\,\mathrm{diag}\{u_i(t)^2\})^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{\lambda}(t)\\&=-\{\boldsymbol{Q}^{-1}+\boldsymbol{Q}^{-1}\boldsymbol{R}(\boldsymbol{I}+\mathrm{diag}\{u_i(t)^2\}\boldsymbol{Q}^{-1}\boldsymbol{R})^{-1}\mathrm{diag}\{u_i(t)^2\}\boldsymbol{Q}^{-1}\}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{\lambda}(t).\end{aligned} \tag{6.25}$$

Based on Eqs. (2.5), (6.18), (6.22), (6.24) and (6.25), we have:

$$\{\boldsymbol{S}'(t)+\boldsymbol{P}+\boldsymbol{S}(t)\boldsymbol{A}+\boldsymbol{A}^{\mathrm{T}}\boldsymbol{S}(t)-\boldsymbol{S}(t)\boldsymbol{B}\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{S}(t)\}\boldsymbol{x}(t)+\boldsymbol{S}(t)\boldsymbol{D}w(t)+\xi'(t)$$
$$-(\boldsymbol{A}^{\mathrm{T}}-\boldsymbol{S}(t)\boldsymbol{B}\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}})\xi(t)-\boldsymbol{S}(t)\boldsymbol{B}\boldsymbol{Q}^{-1}\boldsymbol{R}(\boldsymbol{I}+\mathrm{diag}\{u_i(t)^2\}\boldsymbol{Q}^{-1}\boldsymbol{R})^{-1}\,\mathrm{diag}\{u_i(t)^2\}\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}(\boldsymbol{S}(t)\boldsymbol{x}(t)+\xi(t)). \tag{6.26}$$

Thus, when $\boldsymbol{S}(t)$ and its end condition are given by Eqs. (4.4) and (4.6), respectively, $\xi(t)$ represents higher-order and excitation-dependent terms of $\boldsymbol{\lambda}(t)$.

6.3. Optimization for variable element

Next, let us assume a structure with variable elements. The structural dynamics are expressed by:

$$\boldsymbol{x}'(t)=\boldsymbol{A}\boldsymbol{x}(t)+\left(\sum_j g_j(t)\boldsymbol{U}_j\boldsymbol{U}_j^{\mathrm{T}}\right)\boldsymbol{x}(t)+\boldsymbol{D}w(t). \tag{6.27}$$

where $\boldsymbol{U}_j\in\mathbb{R}^{2n}$ indicates the locations where a variable element is placed. For the VSE, 1 or -1 are assigned in the lower half components of $\boldsymbol{U}_j$. For the VDE, 1 or -1 are assigned in the upper half components of $\boldsymbol{U}_j$.

Let us obtain the optimal control forces that minimize the cost function J:

$$J=F_1(\boldsymbol{x}(t_1),t_1)+\int_{t_0}^{t_1}F(\boldsymbol{x}(t),g_j(t),t)\,\mathrm{d}t. \tag{6.28}$$

Assuming $\boldsymbol{\lambda}(t)\in\mathbb{R}^{2n}$, the Hamiltonian is defined as:

$$H=F(\boldsymbol{x}(t),g_j(t),t)+\boldsymbol{\lambda}(t)^{\mathrm{T}}\left\{\boldsymbol{A}\boldsymbol{x}(t)+\left(\sum_j g_j(t)\boldsymbol{U}_j\boldsymbol{U}_j^{\mathrm{T}}\right)\boldsymbol{x}(t)+\boldsymbol{D}w(t)\right\}. \tag{6.29}$$

Then, for the optimal values,

$$\partial H/\partial\boldsymbol{x}=\partial F(\boldsymbol{x}(t),g_j(t),t)/\partial\boldsymbol{x}+\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\lambda}(t)+\left(\sum_j g_j(t)\boldsymbol{U}_j\boldsymbol{U}_j^{\mathrm{T}}\right)\boldsymbol{\lambda}(t)=-\boldsymbol{\lambda}'(t), \tag{6.30}$$

$$\partial H/\partial g_j=\partial F(\boldsymbol{x}(t),g_j(t),t)/\partial g_j+\boldsymbol{\lambda}(t)^{\mathrm{T}}\boldsymbol{U}_j\boldsymbol{U}_j^{\mathrm{T}}\boldsymbol{x}(t)=0. \tag{6.31}$$

For $F(\boldsymbol{x}(t))=\frac{1}{2}\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(t)+\left(\sum_j\frac{1}{2}g_j(t)Q_jg_j(t)\right)$ and $F_1(\boldsymbol{x}(t))=\frac{1}{2}\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}_1\boldsymbol{x}(t)$,

$$g_j(t)=-Q_j^{-1}\boldsymbol{\lambda}(t)^{\mathrm{T}}\boldsymbol{U}_j\boldsymbol{U}_j^{\mathrm{T}}\boldsymbol{x}(t), \tag{6.32}$$

$$\boldsymbol{x}'(t)=\boldsymbol{A}\boldsymbol{x}(t)+\left(\sum_j-Q_j^{-1}\boldsymbol{\lambda}(t)^{\mathrm{T}}\boldsymbol{U}_j\boldsymbol{U}_j^{\mathrm{T}}\boldsymbol{x}(t)\boldsymbol{U}_j\boldsymbol{U}_j^{\mathrm{T}}\right)\boldsymbol{x}(t)+\boldsymbol{D}w(t), \tag{6.33}$$

$$\boldsymbol{\lambda}'(t)=-\boldsymbol{P}\boldsymbol{x}(t)-\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\lambda}(t)-\left(\sum_j-Q_j^{-1}\boldsymbol{\lambda}(t)^{\mathrm{T}}\boldsymbol{U}_j\boldsymbol{U}_j^{\mathrm{T}}\boldsymbol{x}(t)\boldsymbol{U}_j\boldsymbol{U}_j^{\mathrm{T}}\right)\boldsymbol{\lambda}(t). \tag{6.34}$$

The initial condition for $\boldsymbol{x}(t_0)$ and the end conditions for $\boldsymbol{\lambda}(t_1)$ are given by:

$$\boldsymbol{x}(t_0)=\boldsymbol{x}_0,\qquad\boldsymbol{\lambda}(t_1)=\partial F_1(\boldsymbol{x}(t_1),t_1)/\partial\boldsymbol{x}=\boldsymbol{P}_1\boldsymbol{x}(t_1). \tag{6.35}$$

To obtain the optimal solution, we must solve the above nonlinear differential equations under two-point boundary conditions.

When neglecting the excitation influence and using the approximation given by $\boldsymbol{\lambda}(t) \sim \boldsymbol{S}(t)\boldsymbol{x}(t)$, we have:

$$g_j(t) = -Q_j^{-1}\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{S}^{\mathrm{T}}(t)\boldsymbol{U}_j\boldsymbol{U}_j^{\mathrm{T}}\boldsymbol{x}(t). \tag{6.36}$$

And, $\boldsymbol{S}(t)$ is given by:

$$\boldsymbol{S}'(t) + \boldsymbol{S}(t)\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{S}(t) + \boldsymbol{P} = 0, \tag{6.37}$$

$$\boldsymbol{S}(t_1) = \boldsymbol{P}_1. \tag{6.38}$$

To consider excitation influence, we may assume a state equation model for excitation information.

6.4. Optimization for variable damping element with auxiliary stiffness element

As reviewed in Section 2.6, damping elements are often installed in a structure via ASEs. Thus, VDEs accompanying ASEs compose NMW elements. Let us introduce a control law for such VDEs, assuming a structure equipped with m pairs of them. Let one pair of ASE stiffness and VDE damping coefficients be g_j and $\eta_j(t)$, respectively. Then, the dynamics for a structure with these VDEs and ASEs are expressed by:

$$\boldsymbol{My}''(t) + \boldsymbol{Cy}'(t) + \boldsymbol{Ky}(t) + \boldsymbol{Uu}(t) = -\boldsymbol{MV}w(t), \tag{6.39}$$

$$\boldsymbol{u}'(t) = \boldsymbol{GU}^{\mathrm{T}}\boldsymbol{y}'(t) - \boldsymbol{E}_{\mathrm{L}}(t)\boldsymbol{u}(t) + \boldsymbol{e}(t) \circ \boldsymbol{u}(t), \tag{6.40}$$

where $\boldsymbol{U}$ indicates the DOFs where the controlled force acts, $\boldsymbol{u}(t)$ represents controlled forces in the NMW elements, $\boldsymbol{G} = \mathrm{diag}\{g_j\}$, $\boldsymbol{E}_{\mathrm{L}} = \mathrm{diag}\{e_{Lj}\}$, $\boldsymbol{e}(t) = \{e_j(t)\}$, $\eta_j(t) = g_j/(e_{Lj} - e_j(t))$, and $\boldsymbol{e}(t) \circ \boldsymbol{u}(t) = \{e_j u_j\}$.Or, we can write:

$$\boldsymbol{x}'(t) = \boldsymbol{Ax}(t) + \boldsymbol{Bu}(t) + \boldsymbol{D}w(t), \tag{6.41}$$

$$\boldsymbol{u}'(t) = \boldsymbol{GJ}^{\mathrm{T}}\boldsymbol{x}(t) - \boldsymbol{E}_{\mathrm{L}}\boldsymbol{u}(t) + \boldsymbol{e}(t) \circ \boldsymbol{u}(t), \tag{6.42}$$

where $\boldsymbol{J} = [\boldsymbol{U}^{\mathrm{T}} \quad \boldsymbol{0}]$.

Because the controlled force is not explicitly defined by the function of the state values, but its dynamics are defined by a differential equation, the control for the VDE with the ASE is one of indirect control methods.

Then, let us obtain the optimal controlled force that minimizes the cost function J:

$$J = \boldsymbol{x}(t_1)^{\mathrm{T}}\boldsymbol{P}_1\boldsymbol{x}(t_1) + \int_{t_0}^{t_1} \frac{1}{2}(\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{Px}(t) + \boldsymbol{u}(t)^{\mathrm{T}}\boldsymbol{Qu}(t) + \boldsymbol{e}(t)^{\mathrm{T}}\boldsymbol{Re}(t))\,\mathrm{d}t. \tag{6.43}$$

Assuming $\boldsymbol{\lambda}(t) \in \mathbb{R}^{2n}$ and $\boldsymbol{\varphi}(t) \in \mathbb{R}^m$, the Hamiltonian is defined as:

$$H = \tfrac{1}{2}(\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{Px}(t) + \boldsymbol{u}(t)^{\mathrm{T}}\boldsymbol{Qu}(t) + \boldsymbol{e}(t)^{\mathrm{T}}\boldsymbol{Re}(t)) + \boldsymbol{\lambda}(t)^{\mathrm{T}}(\boldsymbol{Ax}(t) + \boldsymbol{Bu}(t) + \boldsymbol{D}w(t)) + \boldsymbol{\varphi}(t)^{\mathrm{T}}(\boldsymbol{GJ}^{\mathrm{T}}\boldsymbol{x}(t) - \boldsymbol{E}_{\mathrm{L}}\boldsymbol{u}(t) + \boldsymbol{e}(t) \circ \boldsymbol{u}(t)). \tag{6.44}$$

Then, the optimal values satisfy:

$$-\boldsymbol{\lambda}'(t) = \partial H/\partial \boldsymbol{x} = \boldsymbol{Px}(t) + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\lambda}(t) + \boldsymbol{JG\varphi}(t), \tag{6.45}$$

$$-\boldsymbol{\varphi}'(t) = \partial H/\partial \boldsymbol{u} = \boldsymbol{Qu}(t) + \boldsymbol{B}^{\mathrm{T}}\boldsymbol{\lambda}(t) - \boldsymbol{E}_{\mathrm{L}}\boldsymbol{\varphi}(t) + \boldsymbol{e}(t) \circ \boldsymbol{\varphi}(t), \tag{6.46}$$

$$0 = \partial H/\partial \boldsymbol{e} = \boldsymbol{Re}(t) + \boldsymbol{u}(t) \circ \boldsymbol{\varphi}(t). \tag{6.47}$$

Hence,

$$\boldsymbol{e}(t) = -\boldsymbol{R}^{-1}\boldsymbol{u}(t) \circ \boldsymbol{\varphi}(t), \tag{6.48}$$

$$\boldsymbol{\varphi}'(t) = -\boldsymbol{Q}\boldsymbol{u}(t) - \boldsymbol{B}^{\mathrm{T}}\boldsymbol{\lambda}(t) + \boldsymbol{E}_{\mathrm{L}}\boldsymbol{\varphi}(t) + \boldsymbol{R}^{-1}\boldsymbol{u}(t) \circ \boldsymbol{\varphi}(t) \circ \boldsymbol{\varphi}(t), \tag{6.49}$$

$$\boldsymbol{u}'(t) = \boldsymbol{G}\boldsymbol{J}^{\mathrm{T}}\boldsymbol{x}(t) - \boldsymbol{E}_{\mathrm{L}}\boldsymbol{u}(t) - \boldsymbol{R}^{-1}\boldsymbol{u}(t) \circ \boldsymbol{u}(t) \circ \boldsymbol{\varphi}(t). \tag{6.50}$$

The initial condition for $\boldsymbol{x}(t_0)$ and the end conditions for $\boldsymbol{\lambda}(t_1)$ and $\boldsymbol{\varphi}(t_1)$ are given by:

$$\boldsymbol{x}(t_0) = \boldsymbol{x}_0, \quad \boldsymbol{\lambda}(t_1) = \boldsymbol{P}\boldsymbol{x}(t_1), \quad \boldsymbol{\varphi}(t_1) = 0. \tag{6.51}$$

Solving Eqs. (6.41), (6.42), (6.49) and (6.50) under two-point boundary conditions provides the optimal control law.

To consider excitation influence, let us assume a state equation model for excitation provided by Eqs. (5.1) and (5.2). Then, we can write Eqs. (6.41), (6.42), (6.49) and (6.50) as:

$$\boldsymbol{\chi}'(t) = \overline{\boldsymbol{A}}\boldsymbol{\chi}(t) + \overline{\boldsymbol{D}}(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{v}(t) + \zeta(t)) - \overline{\boldsymbol{B}}(\boldsymbol{u}(t) \circ \boldsymbol{u}(t) \circ \boldsymbol{\varphi}(t)), \tag{6.52}$$

$$\boldsymbol{\mu}'(t) = -\overline{\boldsymbol{P}}\boldsymbol{\chi}(t) - \overline{\boldsymbol{A}}^{\mathrm{T}}\boldsymbol{\mu}(t) + \overline{\boldsymbol{B}}(\boldsymbol{u}(t) \circ \boldsymbol{\varphi}(t) \circ \boldsymbol{\varphi}(t)), \tag{6.53}$$

where

$$\boldsymbol{\chi}(t) = \begin{Bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{u}(t) \end{Bmatrix}, \qquad \overline{\boldsymbol{A}} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{G}\boldsymbol{J}^{\mathrm{T}} & -\boldsymbol{E}_{\mathrm{L}} \end{bmatrix}, \qquad \overline{\boldsymbol{P}} = \begin{bmatrix} \boldsymbol{P} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{Q} \end{bmatrix}, \qquad \overline{\boldsymbol{D}} = \begin{bmatrix} \boldsymbol{D} \\ \boldsymbol{0} \end{bmatrix}, \qquad \overline{\boldsymbol{B}} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{R}^{-1} \end{bmatrix},$$

i.e., $\boldsymbol{\chi}(t)$, $\boldsymbol{\mu}(t)$ and $\overline{\boldsymbol{D}} \in \mathbb{R}^{2n+m}$; $\overline{\boldsymbol{A}}$ and $\overline{\boldsymbol{P}} \in \mathbb{R}^{(2n+m)\times(2n+m)}$; $\overline{\boldsymbol{B}} \in \mathbb{R}^{(2n+m)\times m}$.

Let us assume:

$$\boldsymbol{\mu}(t) = \overline{\boldsymbol{S}}(t)\boldsymbol{\chi}(t) + \overline{\boldsymbol{Y}}(t)\boldsymbol{v}(t) + \boldsymbol{\xi}(t), \tag{6.54}$$

where $\overline{\boldsymbol{S}}(t) \in \mathbb{R}^{(2n+m)\times(2n+m)}$, $\overline{\boldsymbol{Y}}(t) \in \mathbb{R}^{(2n+m)\times q}$ and $\boldsymbol{\xi}(t) \in \mathbb{R}^{2n+m}$.

Then, we have:

$$\begin{aligned}&\{\overline{\boldsymbol{S}}'(t) + \overline{\boldsymbol{S}}(t)\overline{\boldsymbol{A}} + \overline{\boldsymbol{A}}^{\mathrm{T}}\overline{\boldsymbol{S}}(t) + \overline{\boldsymbol{P}}\}\boldsymbol{\chi}(t) + \{\overline{\boldsymbol{Y}}'(t) + \overline{\boldsymbol{S}}(t)\overline{\boldsymbol{D}}\boldsymbol{F}^{\mathrm{T}} + \overline{\boldsymbol{A}}^{\mathrm{T}}\overline{\boldsymbol{Y}}(t) + \overline{\boldsymbol{Y}}(t)\boldsymbol{H}\}\boldsymbol{v}(t) \\ &\quad -\overline{\boldsymbol{S}}(t)\overline{\boldsymbol{B}}(\boldsymbol{u}(t) \circ \boldsymbol{u}(t) \circ \boldsymbol{\varphi}(t)) - \overline{\boldsymbol{B}}(\boldsymbol{u}(t) \circ \boldsymbol{u}(t) \circ \boldsymbol{\varphi}(t)) + (\overline{\boldsymbol{S}}(t)\overline{\boldsymbol{D}} + \overline{\boldsymbol{Y}}(t)\boldsymbol{L})\zeta(t) + \boldsymbol{\xi}'(t) + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\xi}(t) = 0.\end{aligned} \tag{6.55}$$

Let $\overline{\boldsymbol{S}}(t)$ and $\overline{\boldsymbol{Y}}(t)$ follow:

$$\overline{\boldsymbol{S}}'(t) + \overline{\boldsymbol{S}}(t)\overline{\boldsymbol{A}} + \overline{\boldsymbol{A}}^{\mathrm{T}}\overline{\boldsymbol{S}}(t) + \overline{\boldsymbol{P}} = 0, \tag{6.56}$$

$$\overline{\boldsymbol{Y}}'(t) + \overline{\boldsymbol{S}}(t)\overline{\boldsymbol{D}}\boldsymbol{F}^{\mathrm{T}} + \overline{\boldsymbol{A}}^{\mathrm{T}}\overline{\boldsymbol{Y}}(t) + \overline{\boldsymbol{Y}}(t)\boldsymbol{H} = 0, \tag{6.57}$$

$$\overline{\boldsymbol{S}}(t_1) = \boldsymbol{P}_1, \quad \overline{\boldsymbol{Y}}(t_1) = 0. \tag{6.58}$$

Then, $\overline{\boldsymbol{S}}(t)\boldsymbol{\chi}(t) + \overline{\boldsymbol{Y}}(t)\boldsymbol{v}(t)$ provides the first-order approximation of $\boldsymbol{\mu}(t)$, and $\boldsymbol{\xi}(t)$ owes higher-order terms.

Or, more simply, if we neglect excitation influence and assume $\boldsymbol{\varphi}(t) \sim \overline{\boldsymbol{S}}\boldsymbol{u}(t)$, $\overline{\boldsymbol{S}} \in \mathbb{R}^{m\times m}$, we have:

$$\boldsymbol{u}'(t) = \boldsymbol{G}\boldsymbol{J}^{\mathrm{T}}\boldsymbol{x}(t) - \boldsymbol{E}_{\mathrm{L}}\boldsymbol{u}(t) - \boldsymbol{R}^{-1}\overline{\boldsymbol{S}}\boldsymbol{u}(t) \circ \boldsymbol{u}(t) \circ \boldsymbol{u}(t), \tag{6.59}$$

$$\boldsymbol{e}(t) = -\boldsymbol{R}^{-1}\overline{\boldsymbol{S}}\boldsymbol{u}(t) \circ \boldsymbol{u}(t). \tag{6.60}$$

Then, $\boldsymbol{e}(t)$ should be decreased for large-amplitude control forces for $\overline{\boldsymbol{S}} > 0$.

7. Stability conditions for nonlinear control

This section reviews sufficient conditions for stability for direct control and indirect control. That is, direct control explicitly expresses a control force by state values, while indirect control defines control force dynamics by a differential equation.

7.1. Direct nonlinear control for free vibration

First, let us neglect an excitation, and then a dynamics of a model is given by:

$$\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t). \tag{7.1}$$

where $\boldsymbol{A}$ is a Hurwitz matrix, i.e., real parts of all eigenvalues of $\boldsymbol{A}$ are negative. Then, $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} < 0$ for $\boldsymbol{P} > 0$.

As direct nonlinear control, let us assume control forces expressed by:

$$\boldsymbol{u}(t) = -\boldsymbol{\Theta}(\boldsymbol{G}\boldsymbol{x}(t)). \tag{7.2}$$

Condition A (Popov criterion**).** *Assume control forces satisfying*

$$\boldsymbol{\Theta}(\boldsymbol{G}\boldsymbol{x}(t))^{\mathrm{T}}[\boldsymbol{\Theta}(\boldsymbol{G}\boldsymbol{x}(t)) - \boldsymbol{Q}\boldsymbol{G}\boldsymbol{x}(t)] \leqslant 0, \tag{7.3}$$

where

$$\boldsymbol{Q} = \mathrm{diag}\{..q_j..\}. \tag{7.4}$$

That is, for the jth column of $\boldsymbol{\Theta}_j(\boldsymbol{y}(t))$ with $\boldsymbol{y}(t) = \boldsymbol{G}\boldsymbol{x}(t)$,

$$\boldsymbol{\Theta}_j(\boldsymbol{y}(t))^{\mathrm{T}}[\boldsymbol{\Theta}_j(\boldsymbol{y}(t)) - q_j\boldsymbol{y}(t)] \leqslant 0. \tag{7.5}$$

Then, a system is stable if there exists $\boldsymbol{P} > 0$ and $\gamma > 0$, such that:

$$\begin{aligned} &\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} = -\boldsymbol{L}^{\mathrm{T}}\boldsymbol{L} - \varepsilon\boldsymbol{P}, \\ &\boldsymbol{P}\boldsymbol{B} - \gamma\boldsymbol{A}^{\mathrm{T}}\boldsymbol{G}^{\mathrm{T}}\boldsymbol{Q} = \boldsymbol{G}^{\mathrm{T}}\boldsymbol{Q} - \boldsymbol{L}^{\mathrm{T}}\boldsymbol{W}, \\ &\gamma(\boldsymbol{Q}\boldsymbol{G}\boldsymbol{B} + \boldsymbol{B}^{\mathrm{T}}\boldsymbol{G}^{\mathrm{T}}\boldsymbol{Q}) = \boldsymbol{W}^{\mathrm{T}}\boldsymbol{W} - 2\boldsymbol{I}. \end{aligned} \tag{7.6}$$

Proof. Assume a Lyapunov function candidate defined by:

$$V = \boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(t) + 2\gamma\int_0^{\boldsymbol{G}\boldsymbol{x}(t)} \boldsymbol{\Theta}(\sigma)^{\mathrm{T}}\boldsymbol{Q}\,\mathrm{d}\sigma. \tag{7.7}$$

where $\boldsymbol{P} = \boldsymbol{P}^{\mathrm{T}} > 0$ and $\boldsymbol{Q} = \boldsymbol{Q}^{\mathrm{T}} > 0$.
Then,

$$\begin{aligned}V' &= \boldsymbol{x}'(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(t) + \boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}'(t) + 2\gamma\boldsymbol{\Theta}(\boldsymbol{G}\boldsymbol{x}(t))^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{G}\boldsymbol{x}'(t)\\ &= \boldsymbol{x}(t)^{\mathrm{T}}[\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}]\boldsymbol{x}(t) - 2\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{B}\Theta(\boldsymbol{G}\boldsymbol{x}(t)) + 2\gamma\boldsymbol{\Theta}(\boldsymbol{G}\boldsymbol{x}(t))^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{G}[\boldsymbol{A}\boldsymbol{x}(t) - \boldsymbol{B}\Theta(\boldsymbol{G}\boldsymbol{x}(t))]\\ &= \boldsymbol{x}(t)^{\mathrm{T}}[\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}]\boldsymbol{x}(t) - 2\boldsymbol{x}(t)^{\mathrm{T}}[\boldsymbol{P}\boldsymbol{B} - \gamma\boldsymbol{A}^{\mathrm{T}}\boldsymbol{G}^{\mathrm{T}}\boldsymbol{Q}]\boldsymbol{\Theta}(\boldsymbol{G}\boldsymbol{x}(t)) - 2\gamma\boldsymbol{\Theta}(\boldsymbol{G}\boldsymbol{x}(t))^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{G}\boldsymbol{B}\Theta(\boldsymbol{G}\boldsymbol{x}(t))\\ &= -\boldsymbol{x}(t)^{\mathrm{T}}[\boldsymbol{L}^{\mathrm{T}}\boldsymbol{L} + \varepsilon\boldsymbol{P}]\boldsymbol{x}(t) - 2\boldsymbol{x}(t)^{\mathrm{T}}[\boldsymbol{G}^{\mathrm{T}}\boldsymbol{Q} - \boldsymbol{L}^{\mathrm{T}}\boldsymbol{W}]\boldsymbol{\Theta}(\boldsymbol{G}\boldsymbol{x}(t)) - \boldsymbol{\Theta}(\boldsymbol{G}\boldsymbol{x}(t))^{\mathrm{T}}[\boldsymbol{W}^{\mathrm{T}}\boldsymbol{W} - 2\boldsymbol{I}]\boldsymbol{\Theta}(\boldsymbol{G}\boldsymbol{x}(t))\\ &= -\varepsilon\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{x}(t) - [\boldsymbol{L}\boldsymbol{x}(t) - \boldsymbol{W}\Theta(\boldsymbol{G}\boldsymbol{x}(t))]^{\mathrm{T}}[\boldsymbol{L}\boldsymbol{x}(t) - \boldsymbol{W}\Theta(\boldsymbol{G}\boldsymbol{x}(t))] - \boldsymbol{\Theta}(\boldsymbol{G}\boldsymbol{x}(t))^{\mathrm{T}}[\boldsymbol{\Theta}(\boldsymbol{G}\boldsymbol{x}(t)) - \boldsymbol{Q}\boldsymbol{G}(x)]\\ &< 0.\end{aligned} \tag{7.8}$$

Since (i) $V = 0$ only if $\boldsymbol{x}(t) = 0$ and (ii) $V > 0$ and $V' < 0$ unless $\boldsymbol{x}(t) = 0$, the system becomes stable.

This condition is equivalent to the existence of the positive real $\boldsymbol{z}(s)$ defined by:

$$\boldsymbol{z}(s) = \boldsymbol{I} + (1 + \gamma s)\boldsymbol{Q}\boldsymbol{G}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}. \quad \square \tag{7.9}$$

Example A: A control system is stable if the *j*th control force given by:

$$u_j(t) = -q_j\,\mathrm{sat}(x_k(t) - x_l(t)), \tag{7.10}$$

where $\mathrm{sat}(x(t)) = 1$ for $|x(t)| \geqslant 1$ and $\mathrm{sat}(x(t)) = x(t)$ for $|x(t)| < 1$, and k and l are the DOFs where the *j*th control force act.

Condition B (Circle Criterion**).** *Assume that control forces are given by*:

$$\boldsymbol{\Theta}(\boldsymbol{G}\boldsymbol{x}(t))^{\mathrm{T}}[\boldsymbol{\Theta}(\boldsymbol{G}\boldsymbol{x}(t)) - \boldsymbol{Q}\boldsymbol{G}\boldsymbol{x}(t)] \leqslant 0, \tag{7.11}$$

where $\boldsymbol{Q} > 0$.

Then, a system is stable if there exists $\boldsymbol{P} > 0$ *such that*:

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} = -\boldsymbol{L}^{\mathrm{T}}\boldsymbol{L} - \varepsilon\boldsymbol{P}, \qquad \boldsymbol{P}\boldsymbol{B} = \boldsymbol{G}^{\mathrm{T}}\boldsymbol{Q} - \sqrt{2}\boldsymbol{L}^{\mathrm{T}} \quad \text{for } \varepsilon > 0. \tag{7.12}$$

Proof. Assuming a Lyapunov function candidate defined by:

$$V = \boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(t), \tag{7.13}$$

$$\begin{aligned}V' &= \boldsymbol{x}'(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(t) + \boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(t)\\ &= [\boldsymbol{A}\boldsymbol{x}(t) - \boldsymbol{B}\Theta(\boldsymbol{G}\boldsymbol{x}(t))]^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(t) + \boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}[\boldsymbol{A}\boldsymbol{x}(t) - \boldsymbol{B}\Theta(\boldsymbol{G}\boldsymbol{x}(t))]\\ &= \boldsymbol{x}(t)^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A})\boldsymbol{x}(t) - 2\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{B}\Theta(\boldsymbol{G}\boldsymbol{x}(t))\\ &= -\boldsymbol{x}(t)^{\mathrm{T}}(\boldsymbol{L}^{\mathrm{T}}\boldsymbol{L} + \varepsilon\boldsymbol{P})\boldsymbol{x}(t) - 2\boldsymbol{x}(t)^{\mathrm{T}}(\boldsymbol{G}^{\mathrm{T}}\boldsymbol{Q} - \sqrt{2}\boldsymbol{L}^{\mathrm{T}})\boldsymbol{\Theta}(\boldsymbol{G}\boldsymbol{x}(t))\\ &= -\varepsilon\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(t) - [\boldsymbol{L}\boldsymbol{x}(t) - \sqrt{2}\boldsymbol{\Theta}(\boldsymbol{G}\boldsymbol{x}(t))]^{\mathrm{T}}[\boldsymbol{L}\boldsymbol{x}(t) - \sqrt{2}\boldsymbol{\Theta}(\boldsymbol{G}\boldsymbol{x}(t))] + 2\boldsymbol{\Theta}(\boldsymbol{G}\boldsymbol{x}(t))^{\mathrm{T}}[\boldsymbol{\Theta}(\boldsymbol{G}\boldsymbol{x}(t))\\ &\qquad - \boldsymbol{Q}\boldsymbol{G}\boldsymbol{x}(t)] < 0.\end{aligned} \tag{7.14}$$

Since (i) $V = 0$ only if $x = 0$ and (ii) $V > 0$ and $V' < 0$ unless $x = 0$, the system becomes stable.

This condition is equivalent to the existence of positive real $\boldsymbol{z}(s)$ such that:

$$\boldsymbol{z}(s) = \boldsymbol{I} + \boldsymbol{Q}\boldsymbol{G}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}. \quad \square \tag{7.15}$$

Example B: Suppose control forces given by:

$$\boldsymbol{u}(t) = -\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}\,\mathrm{sat}(\boldsymbol{x}(t)), \tag{7.16}$$

where $\mathrm{sat}(\boldsymbol{x}(t)) = \{..\mathrm{sat}(x_i(t))..\}^{\mathrm{T}}$ and $\boldsymbol{Q} = \boldsymbol{Q}^{\mathrm{T}} > 0$.Then, the system is stable since:

$$\begin{aligned}&[\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}\,\mathrm{sat}(\boldsymbol{x}(t))]^{\mathrm{T}}[\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}\,\mathrm{sat}(\boldsymbol{x}(t)) - \boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{x}(t)]\\ &\quad = \mathrm{sat}(\boldsymbol{x}(t))^{\mathrm{T}}\boldsymbol{B}\boldsymbol{Q}^{-1}\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}[\mathrm{sat}(\boldsymbol{x}(t)) - \boldsymbol{x}(t)] \leqslant 0.\end{aligned} \tag{7.17}$$

7.2. Indirect nonlinear control for free vibration

Next, for the model defined by Eq. (7.1), let us assume a control force defined by the differential equation:

$$\boldsymbol{u}'(t) = \boldsymbol{\Theta}(\boldsymbol{x}(t), \boldsymbol{u}(t)). \tag{7.18}$$

Condition C (by Lefshetz**).** *The system is stable if* $\boldsymbol{u}'(t)$ *is given by*:

$$\begin{aligned}&\boldsymbol{u}'(t) = \boldsymbol{\Theta}(\boldsymbol{G}\boldsymbol{x}(t) - \boldsymbol{E}\boldsymbol{u}(t)), \boldsymbol{E} > 0 \quad \text{and} \begin{bmatrix} (\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}) & (\boldsymbol{P}\boldsymbol{B} + \boldsymbol{G}^{\mathrm{T}}\boldsymbol{Q}) \\ (\boldsymbol{P}\boldsymbol{B} + \boldsymbol{G}^{\mathrm{T}}\boldsymbol{Q})^{\mathrm{T}} & -2\boldsymbol{Q}\boldsymbol{E} \end{bmatrix} < 0, \quad \text{for } \boldsymbol{P} = \boldsymbol{P}^{\mathrm{T}} > 0\\ &\quad \boldsymbol{Q} = \boldsymbol{Q}^{\mathrm{T}} > 0.\end{aligned} \tag{7.19}$$

Proof. Let us first transform $\boldsymbol{x}(t)$ and $\boldsymbol{u}(t)$ to $\boldsymbol{q}(t)$ and $\boldsymbol{v}(t)$ by:

$$\boldsymbol{q}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \qquad \boldsymbol{v}(t) = \boldsymbol{G}\boldsymbol{x}(t) - \boldsymbol{E}\boldsymbol{u}(t), \tag{7.20}$$

where

$$\begin{vmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{G} & -\boldsymbol{E} \end{vmatrix} \neq 0. \tag{7.21}$$

Then, we have:

$$\boldsymbol{q}'(t) = \boldsymbol{A}\boldsymbol{q}(t) + \boldsymbol{B}\Theta(\boldsymbol{v}(t)), \qquad \boldsymbol{v}'(t) = \boldsymbol{G}\boldsymbol{q}(t) - \boldsymbol{E}\Theta(\boldsymbol{v}(t)). \tag{7.22}$$

Next, let us assume the Lyapunov function candidate:

$$V = \boldsymbol{q}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{q}(t) + 2\int_0^{\boldsymbol{v}(t)} \Theta(\boldsymbol{\sigma})^{\mathrm{T}}\boldsymbol{Q}\,\mathrm{d}\boldsymbol{\sigma}. \tag{7.23}$$

Then,

$$\begin{aligned}V' &= \boldsymbol{q}'(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{q}(t) + \boldsymbol{q}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{q}'(t) + 2\boldsymbol{\Theta}(\boldsymbol{v}(t))^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{v}'(t)\\ &= \boldsymbol{q}(t)^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A})\boldsymbol{q}(t) + 2\boldsymbol{q}(t)^{\mathrm{T}}(\boldsymbol{P}\boldsymbol{B} + \boldsymbol{G}^{\mathrm{T}}\boldsymbol{Q})\boldsymbol{\Theta}(\boldsymbol{v}(t)) - 2\boldsymbol{\Theta}(\boldsymbol{v}(t))^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{E}\boldsymbol{\Theta}(\boldsymbol{v}(t)) < 0.\end{aligned} \tag{7.24}$$

Since (i) $V = 0$ only for $\{\boldsymbol{x}(t), \boldsymbol{u}(t)\}^{\mathrm{T}} = 0$ and (ii) $V > 0$ and $V' < 0$ unless $\{\boldsymbol{x}(t), \boldsymbol{u}(t)\}^{\mathrm{T}} = 0$, the system is stable. □

However, it is too general to apply this condition to a practical case. Then, more specified conditions are introduced as follows:

Condition D. *In fact, we have an easier problem if all the state values where the control force act are fed back. That is, a system is stable if*:

$$\boldsymbol{u}'(t) = \Theta(\boldsymbol{G}\boldsymbol{x}(t) - \boldsymbol{E}\boldsymbol{u}(t)), \qquad \boldsymbol{G} = -\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}, \quad \boldsymbol{P} > 0, \ \ \boldsymbol{Q} > 0 \ \ \boldsymbol{E} > 0, \tag{7.25}$$

This is because Eq. (7.24) *always holds. The design problem is only to find* $\boldsymbol{P}$, $\boldsymbol{Q}$, $\boldsymbol{E}$ *and the characteristics of* $\Theta(\boldsymbol{G}\boldsymbol{x}(t) - \boldsymbol{E}\boldsymbol{u}(t))$.

Example D: As a simple case for Condition D, let us assume:

$$\boldsymbol{P} = \boldsymbol{I}, \quad \boldsymbol{Q} = \mathrm{diag}\{1/q_j\}, \quad \boldsymbol{E} = \mathrm{diag}\{e_j\}. \tag{7.26}$$

That is, a system is stable when controlled by:

$$u_j'(t) = \{-q_j(x_k(t) - x_l(t)) - e_j u(t)\}^3,$$

where k and l indicates the DOFs where the jth control force act.

Condition E. *The system is stable if*:

$$\boldsymbol{u}'(t) = \boldsymbol{G}\boldsymbol{x}(t) - \Theta(\boldsymbol{u}(t)), \qquad \boldsymbol{G} = -\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}, \qquad \boldsymbol{u}(t)^{\mathrm{T}}\boldsymbol{Q}\Theta(\boldsymbol{u}(t)) > 0. \tag{7.27}$$

Proof. Let us define a Lyapunov function candidate with $\boldsymbol{P} = \boldsymbol{P}^{\mathrm{T}} > 0$ and $\boldsymbol{Q} = \boldsymbol{Q}^{\mathrm{T}} > 0$ as:

$$V = \boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(t) + \boldsymbol{u}(t)^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{u}(t). \tag{7.28}$$

Because (i) $V = 0$ only when $\{\boldsymbol{x}(t), \boldsymbol{u}(t\})^{\mathrm{T}} = 0$, and (ii) $V > 0$ unless $\{\boldsymbol{x}(t), \boldsymbol{u}(t)\}^{\mathrm{T}} = 0$, and (iii)

$$\begin{aligned} V' &= \boldsymbol{x}'(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(t) + \boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}'(t) + \boldsymbol{u}'(t)^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{u}(t) + \boldsymbol{u}(t)^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{u}'(t) \\ &= \boldsymbol{x}(t)^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A})\boldsymbol{x}(t) + 2\boldsymbol{x}(t)^{\mathrm{T}}(\boldsymbol{P}\boldsymbol{B} + \boldsymbol{G}^{\mathrm{T}}\boldsymbol{Q})\boldsymbol{u}(t) - 2\boldsymbol{u}(t)^{\mathrm{T}}\boldsymbol{Q}\Theta(\boldsymbol{u}(t)) < 0, \end{aligned} \tag{7.29}$$

the system is stable.

Example E: As the simplest case that satisfies Condition E, we have:

$$\boldsymbol{P} = \boldsymbol{I}, \quad \boldsymbol{Q} = \mathrm{diag}\{1/q_j\}, \quad \boldsymbol{\Theta}(\boldsymbol{u}(t)) = \mathrm{diag}\{e_j\}\{..u_j(t)^3..\}^{\mathrm{T}}. \tag{7.30}$$

That is,

$$u_j'(t) = -q_j(x_k(t) - x_l(t)) - e_j u_j(t)^3, \tag{7.31}$$

where k and l indicate the DOFs where the jth control force act.

7.3. Direct control with excitation term

Next, let us consider the model under an excitation expressed by Eq. (2.5), and the control force given by Eq. (7.2), where stability for free vibration is warranted by the foregoing conditions, i.e., we have ε such that:

$$\boldsymbol{x}(t)^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A})\boldsymbol{x}(t) + 2\boldsymbol{u}(t)^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(t) < -\varepsilon^2(\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(t) + \boldsymbol{u}(t)^{\mathrm{T}}\boldsymbol{u}(t)). \tag{7.32}$$

Then,

$$\begin{aligned} J &= \boldsymbol{x}(t_1)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(t_1) - \boldsymbol{x}(t_0)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(t_0) = \int_{t_0}^{t_1} [(\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(t))]'\,\mathrm{d}t = \int_{t_0}^{t_1} [\boldsymbol{x}'(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(t) + \boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}'(t)]\,\mathrm{d}t \\ &< \int_{t_0}^{t_1} [-\varepsilon^2\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(t) - \varepsilon^2\boldsymbol{u}(t)^{\mathrm{T}}\boldsymbol{u}(t) + 2w(t)\boldsymbol{D}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(t)]\,\mathrm{d}t \\ &= \int_{t_0}^{t_1} [-(\varepsilon\boldsymbol{x}(t) - \boldsymbol{D}w(t))^{\mathrm{T}}\boldsymbol{P}(\varepsilon\boldsymbol{x}(t) - \boldsymbol{D}w(t)) - \varepsilon^2\boldsymbol{u}(t)^{\mathrm{T}}\boldsymbol{u}(t) + w(t)^2\boldsymbol{D}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{D}]\,\mathrm{d}t \\ &< \boldsymbol{D}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{D}\int_{t_0}^{t_1} w(t)^2\,\mathrm{d}t. \end{aligned} \tag{7.33}$$

Thus, the structural response is bounded as far as the excitation is bounded.

7.4. Indirect control with excitation term

Next, the equivalent results are obtained for indirect control under an excitation as well. For example, let us assume that the control force defined by Condition E as:

$$\boldsymbol{u}'(t) = -\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(t) - \boldsymbol{\Theta}(\boldsymbol{u}(t)), \tag{7.34}$$

and we have ε such that:

$$\boldsymbol{x}(t)^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A})\boldsymbol{x}(t) < -\varepsilon^2\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(t). \tag{7.35}$$

Then,

$$\begin{aligned} J &= \boldsymbol{x}(t_1)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(t_1) - \boldsymbol{x}(t_0)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(t_0) + \boldsymbol{u}(t_1)^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{u}(t_1) - \boldsymbol{u}(t_0)^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{u}(t_0) + 2\int_{t_0}^{t_1} \boldsymbol{u}(t)^{\mathrm{T}}\boldsymbol{Q}\Theta(\boldsymbol{u}(t))\,\mathrm{d}t \\ &= \int_{t_0}^{t_1} [(\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(t) + \boldsymbol{u}(t)^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{u}(t))' + 2\boldsymbol{u}(t)^{\mathrm{T}}\boldsymbol{Q}\Theta(\boldsymbol{u}(t))]\,\mathrm{d}t \\ &= \int_{t_0}^{t_1} [\boldsymbol{x}(t)^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A})\boldsymbol{x}(t) + 2w(t)\boldsymbol{D}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(t)]\,\mathrm{d}t \\ &< \int_{t_0}^{t_1} [-\varepsilon^2\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(t) + 2w(t)\boldsymbol{D}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(t)]\,\mathrm{d}t \\ &= \int_{t_0}^{t_1} [-(\varepsilon\boldsymbol{x}(t) - \boldsymbol{D}w(t))^{\mathrm{T}}\boldsymbol{P}(\varepsilon\boldsymbol{x}(t) - \boldsymbol{D}w(t)) + w(t)^2\boldsymbol{D}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{D}]\,\mathrm{d}t < \boldsymbol{D}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{D}\int_{t_0}^{t_1} w(t)^2\,\mathrm{d}t. \end{aligned} \tag{7.36}$$

Thus, providing an input excitation is bounded, the structural responses and the control force at t_1 and the power of the control force are bounded.

7.5. Control strategy for excitation model

For the control strategies considering the state equation model of excitation information as introduced in Section 5, let us examine their stability.

As shown in Eq. (5.15), we can construct a state equation in augmented space. Because $\widetilde{\boldsymbol{A}}$ is a block diagonal matrix, it can be a Hurwitz matrix if both $\boldsymbol{A}$ and $\boldsymbol{H}$ are Hurwitz. Thus, by substituting $\boldsymbol{z}(t)$, $\zeta(t)$, $\widetilde{\boldsymbol{A}}$, $\widetilde{\boldsymbol{B}}$ and $\widetilde{\boldsymbol{D}}$ for $\boldsymbol{x}(t)$, $w(t)$, $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{D}$ in the foregoing conditions, respectively, the foregoing results are effective for the control strategies considering the state equation model of excitation information as they are.

8. Examples of nonlinear control laws

Following with the stability conditions for direct and indirect control laws, this section introduces examples for direct and indirect nonlinear control laws. As the simplest control laws, nonlinear velocity FB laws and NMW-type control force are introduced.

8.1. Nonlinear velocity feedback laws

Let us assume the nonlinear velocity FB laws, i.e., $u(t, y', y) = -G(y, y')y'$. That is, the control force phases must follow the velocity, but the velocity gains nonlinearly depend on velocity and displacement. For symmetry, the gains should be even functions of velocity and displacement. As shown in Fig. 5, the following are examples of the nonlinear velocity FB laws.

VVD: Gain decreases when the velocity increases as $G(y, y') = a|y'|^{\alpha}$, $-1 < \alpha < 0$. Let us assume a simple case such that $u(t, y', y) = -a|y'|^{1/2}\,\mathrm{sgn}(y')$.

VST: The control force linearly changes at first, but later saturates at $\pm u_0$. This is expressed by $u(t, y', y) = -\min(ey', u_0\,\mathrm{sgn}(y'))$, $e > 0$. That is, gain is constant or zero.

VSW: The control force switches over from u_0 to $-u_0$, or from $-u_0$ to u_0 without linearly changing the range: $u(t, y', y) = -u_0\,\mathrm{sgn}(y')$.

Assuming $m = 1.0$, $\omega = 2\pi$, $h = 0.01$, $a = 1.0$, $e = 10$ and $u_0 = 1.0$, let us obtain responses to impulse and a seismic excitation. Fig. 6 shows time histories of the response velocities and the control forces for impulse assuming $y'_0 = 1.0$. The case VSW shows the residual vibrations. In fact, the cases VVD and VSW satisfy neither Condition A nor Condition B. Fig. 7 shows time histories of the response velocities and the control forces for the seismic excitation, JMA-KOBE, NS component of the Hyogoken–Nanbu earthquake

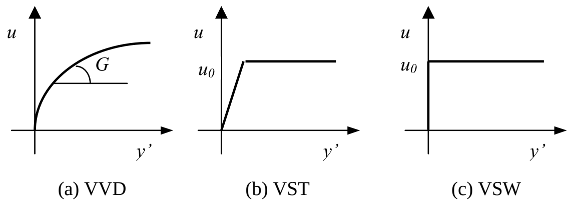

Fig. 5. Assumed nonlinear velocity FB laws.

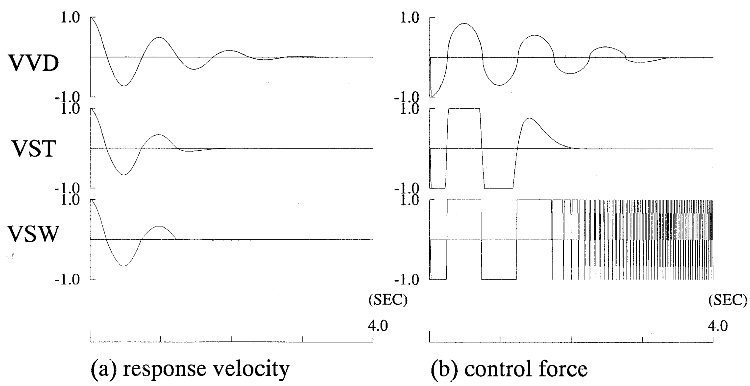

Fig. 6. Response velocities and control forces controlled by nonlinear velocity FB laws for impulse.

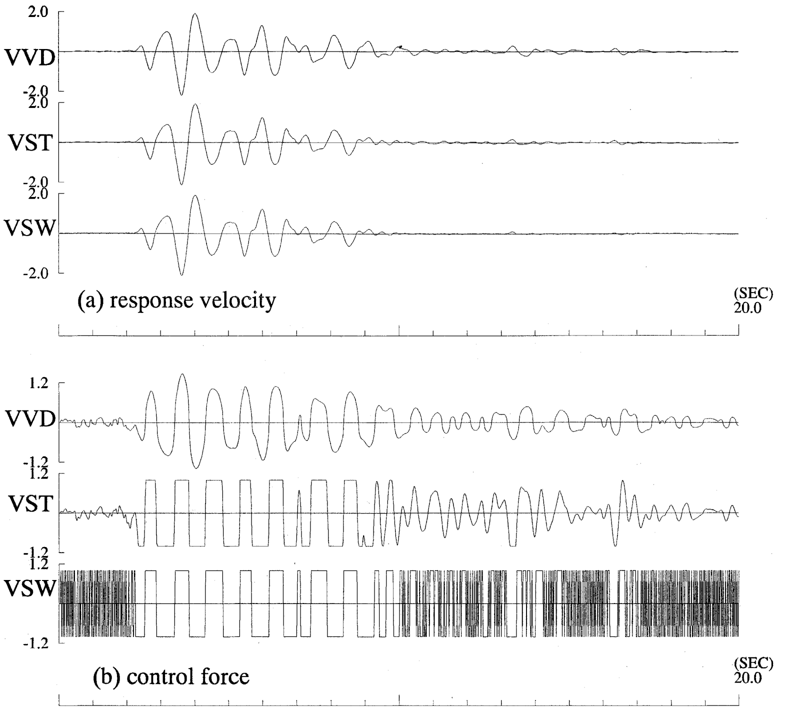

Fig. 7. Response velocities and control forces controlled by nonlinear velocity FD laws for KOBE.

in 1995 (KOBE), assuming that its maximum acceleration is 8.2. We can approximately estimate the control effects by equivalent damping coefficients (Yamada, 1998).

8.2. Nonlinear-Maxwell-type control force

As one of the simplest indirect nonlinear control laws, let the control force be defined by:

$$\boldsymbol{u}'(t) = -\boldsymbol{G}\boldsymbol{U}^{\mathrm{T}}\boldsymbol{y}'(t) - \boldsymbol{E}_{\mathrm{L}}\boldsymbol{u}(t) - \boldsymbol{E}_{\mathrm{N}}\boldsymbol{u}^{3}(t), \tag{8.1}$$

where $\boldsymbol{u}^3(t) = \{u_1(t)^3, \ldots, u_j(t)^3, \ldots, u_m(t)^3\}^{\mathrm{T}}$, i.e., $\boldsymbol{u}^3(t)$ indicates a vector whose components are the third power of each control force. $\boldsymbol{G} \in \mathbb{R}^{m\times m}$, $\boldsymbol{E}_{\mathrm{L}} \in \mathbb{R}^{m\times m}$ and $\boldsymbol{E}_{\mathrm{N}} \in \mathbb{R}^{m\times m}$ represent FB gains given by:

$$\boldsymbol{G} = \mathrm{diag}\{g_j\}, \quad \boldsymbol{E}_{\mathrm{L}} = \mathrm{diag}\{e_{\mathrm{L}j}\}, \quad \boldsymbol{E}_{\mathrm{N}} = \mathrm{diag}\{e_{\mathrm{N}j}\}, \tag{8.2}$$

where diag{ } composes a diagonal matrix of a vector { }. Thus, the control force dynamics depend on the deformation rate between the DOFs and the amount of the control force at time t.

In fact, the control law is provided by Eq. (7.31) and equivalent to the law induced by Eq. (6.59). Thus, it is stable and an approximation of the optimal control when the excitation influence is neglected.

For an SDOF model whose structural dynamics is given by Eq. (2.3), let the control force satisfy:

$$u'(t) = -gy'(t) - e_{\mathrm{L}}u(t) - e_{\mathrm{N}}u(t)^3, \tag{8.3}$$

And, assume $m = 1.0$, $\omega = 2\pi$, $h = 0.01$, $g = 4\pi^2$, $e_{\mathrm{L}} = 5$ and $e_{\mathrm{N}} = 50$. Fig. 8 shows the control force vs. displacement relations for KOBE, whose maximum is 1.0 and 8.2. As shown in the figure, the control force

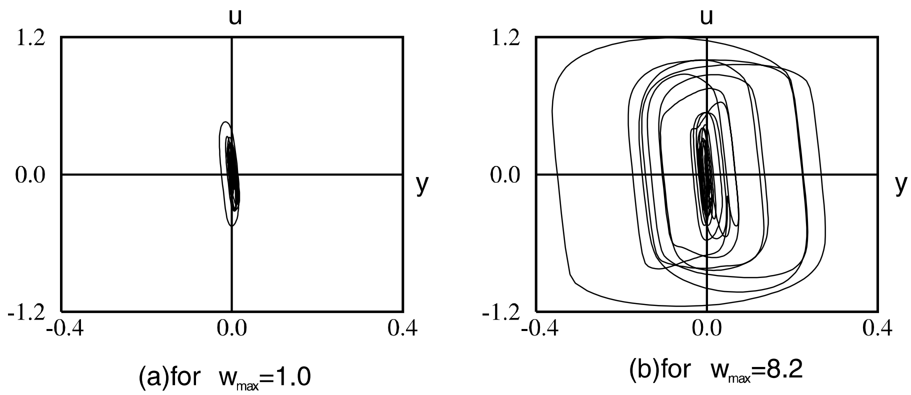

Fig. 8. *u*–*y* relations controlled by NMW-type control force for KOBE.

vs. displacement relations plots elliptical curves for a small excitation, while that for a large excitation draws parallelogram-like curves. Thus, it restrains control force amplitude, making full capacity of the control force.

The performance for MDOF model was discussed by Yamada (2000).

9. Conclusions

Structural control serves us with not only mechanical advantages but also benefits in design and construction. However, a seismic excitation is nonstationary and uncertain and lateral loads amounting to 1 g influence a structure in an extremely large seismic event. To consider control force size and excitation nonstationarity, this paper reviews fundamental dynamics with control forces and introduces control strategies that positively adapt nonlinear effects and excitation influence as follows:

(1) Momentum equations and their modally decomposed forms, state equations and their discrete forms, and energy balance equations are introduced for the structural dynamics with control force under a seismic excitation.

(2) The control effects of linear FB controls, i.e., a control force by gain-constant FB of velocity, deformation and excitation signals were expressed as changes in structural natural period and damping factor and excitation participation factor.

(3) In reviewing linear optimal control laws, the excitation influence in control laws was clarified. That is, the LQRE has a convolution term with future excitation information in addition to the terms provided by the LQR, which neglects excitation influence or assumes a seismic excitation as a white-noise. Furthermore, the control law for the LIE is given only by a convolution term with future excitation information, while the LQRS is given by an instantaneous counter-reaction term and a FB term.

(4) A control strategy using a state equation model for a seismic excitation was analyzed. It was inferred that we have three strategies: (i) to dampen responses to the initial state at each moment, (ii) to activate control force neglecting excitation to a structure, and (iii) to isolate structural natural frequency from the frequency of the excitation sources.

(5) In introducing optimal control problems under constraints, it was inferred that it is difficult to explicitly solve the optimization problems by positively considering control force limit, and the Euler equations for the optimal variable-element control become nonlinear.

(6) Sufficient stability conditions for nonlinear control laws were introduced. It was shown that a control law with information FB at all locations where control forces act can be easily stabilized and that an indirect control can be more straightforwardly stabilized than a direct control law.

(7) As a simple example of nonlinear control laws, nonlinear velocity FB laws and NMW-type control forces are introduced.

However, which strategy is the best depends on the conditions and purposes for each project to which structural control is adopted. That is, we should develop a best control strategy on a case-by-case basis to suite a project's conditions and purposes. In those conditions, the results in this paper would be useful. The authors hope that structural control technologies will be advanced and broadly adopted in our society.

Appendix A. Induction for least quadratic regulator considering excitation influence

Using the minimum principle, the LQRE expressed by Eq. (4.3) is introduced as follows:
Letting $\boldsymbol{\lambda}(t) \in \mathbb{R}^{2n}$, Hamiltonian is defined by:

$$H = \boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(t) + \boldsymbol{u}(t)^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{u}(t) + \boldsymbol{\lambda}(t)^{\mathrm{T}}\{\boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{D}w(t)\}. \tag{A.1}$$

Then, the following equations are satisfied for the optimal values:

$$\boldsymbol{\lambda}'(t) = -\partial H/\partial \boldsymbol{x} = -\boldsymbol{P}\boldsymbol{x}(t) - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\lambda}(t). \tag{A.2}$$

$$0 = \partial H/\partial \boldsymbol{u} = \boldsymbol{Q}\boldsymbol{u}(t) + \boldsymbol{B}^{\mathrm{T}}\boldsymbol{\lambda}(t). \tag{A.3}$$

Thus,

$$\boldsymbol{u}(t) = -\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{\lambda}(t) \tag{A.4}$$

From Eqs. (2.5) and (A.4), we obtain:

$$\boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{D}w(t) - \boldsymbol{B}\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{\lambda}(t). \tag{A.5}$$

Eqs. (A.2) and (A.5) can be written as:

$$\boldsymbol{z}'(t) = \widehat{\boldsymbol{A}}\boldsymbol{z}(t) + \widehat{\boldsymbol{D}}w(t), \tag{A.6}$$

where

$$\boldsymbol{z} = \begin{Bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{\lambda}(t) \end{Bmatrix}, \qquad \widehat{\boldsymbol{A}} = \begin{bmatrix} \boldsymbol{A} & -\boldsymbol{B}\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}} \\ -\boldsymbol{P} & -\boldsymbol{A}^{\mathrm{T}} \end{bmatrix}, \qquad \widehat{\boldsymbol{D}} = \begin{Bmatrix} \boldsymbol{D} \\ \boldsymbol{0} \end{Bmatrix}.$$

The initial and end conditions are

$$\boldsymbol{x}(t_0) = \boldsymbol{x}_0, \quad \boldsymbol{\lambda}(t_1) = \boldsymbol{P}_1\boldsymbol{x}(t_1). \tag{A.7}$$

Thus, we have a two-point boundary problem.

Letting

$$\boldsymbol{\theta}(t,\tau) = \exp(\widehat{\boldsymbol{A}}(t-\tau)) = \begin{bmatrix} \boldsymbol{\theta}_{11}(t,\tau) & \boldsymbol{\theta}_{12}(t,\tau) \\ \boldsymbol{\theta}_{21}(t,\tau) & \boldsymbol{\theta}_{22}(t,\tau) \end{bmatrix},$$

we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\theta}(t,\tau) = \widehat{\boldsymbol{A}}\boldsymbol{\theta} = \begin{bmatrix} \boldsymbol{A} & -\boldsymbol{B}\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}} \\ -\boldsymbol{P} & -\boldsymbol{A}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \boldsymbol{\theta}_{11}(t,\tau) & \boldsymbol{\theta}_{12}(t,\tau) \\ \boldsymbol{\theta}_{21}(t,\tau) & \boldsymbol{\theta}_{22}(t,\tau) \end{bmatrix}, \tag{A.8}$$

where

$$\boldsymbol{\theta}(t,t) = \begin{bmatrix} \boldsymbol{\theta}_{11}(t,t) & \boldsymbol{\theta}_{12}(t,t) \\ \boldsymbol{\theta}_{21}(t,t) & \boldsymbol{\theta}_{22}(t,t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}.$$

Thus, the solution of Eq. (A.6) is expressed by:

$$\boldsymbol{z}(t) = -\boldsymbol{\theta}(t_1,t)\boldsymbol{z}(t_1) - \int_t^{t_1} \boldsymbol{\theta}(t,\tau)\widehat{\boldsymbol{D}}w(\tau)\,\mathrm{d}\tau = \boldsymbol{\theta}(t,t_1)\boldsymbol{z}(t_1) + \int_{t_1}^{t} \boldsymbol{\theta}(t,\tau)\widehat{\boldsymbol{D}}w(\tau)\,\mathrm{d}\tau. \tag{A.9}$$

That is,

$$\begin{Bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{\lambda}(t) \end{Bmatrix} = \begin{bmatrix} \boldsymbol{\theta}_{11}(t,t_1) & \boldsymbol{\theta}_{12}(t,t_1) \\ \boldsymbol{\theta}_{21}(t,t_1) & \boldsymbol{\theta}_{22}(t,t_1) \end{bmatrix} \begin{Bmatrix} \boldsymbol{x}(t_1) \\ \boldsymbol{\lambda}(t_1) \end{Bmatrix} + \int_{t_1}^{t} \begin{bmatrix} \boldsymbol{\theta}_{11}(t,\tau) & \boldsymbol{\theta}_{12}(t,\tau) \\ \boldsymbol{\theta}_{21}(t,\tau) & \boldsymbol{\theta}_{22}(t,\tau) \end{bmatrix} \begin{Bmatrix} \boldsymbol{D} \\ \boldsymbol{0} \end{Bmatrix} w(\tau)\,\mathrm{d}\tau. \tag{A.10}$$

Let us express the above for each component:

$$\boldsymbol{x}(t) = \boldsymbol{\theta}_{11}(t,t_1)\boldsymbol{x}(t_1) + \boldsymbol{\theta}_{12}(t,t_1)\boldsymbol{\lambda}(t_1) + \int_{t_1}^{t} \boldsymbol{\theta}_{11}(t,\tau)\boldsymbol{D}w(\tau)\,\mathrm{d}\tau, \tag{A.11}$$

$$\boldsymbol{\lambda}(t) = \boldsymbol{\theta}_{21}(t,t_1)\boldsymbol{x}(t_1) + \boldsymbol{\theta}_{22}(t,t_1)\boldsymbol{\lambda}(t_1) + \int_{t_1}^{t} \boldsymbol{\theta}_{21}(t,\tau)\boldsymbol{D}w(\tau)\,\mathrm{d}\tau. \tag{A.12}$$

By putting Eq. (A.7) into the above,

$$\boldsymbol{x}(t) = [\boldsymbol{\theta}_{11}(t,t_1) + \boldsymbol{\theta}_{12}(t,t_1)\boldsymbol{P}_1]\boldsymbol{x}(t_1) + \int_{t_1}^{t} \boldsymbol{\theta}_{11}(t,\tau)\boldsymbol{D}w(\tau)\,\mathrm{d}\tau, \tag{A.13}$$

$$\boldsymbol{\lambda}(t) = [\boldsymbol{\theta}_{21}(t,t_1) + \boldsymbol{\theta}_{22}(t,t_1)\boldsymbol{P}_1]\boldsymbol{x}(t_1) + \int_{t_1}^{t} \boldsymbol{\theta}_{21}(t,\tau)\boldsymbol{D}w(\tau)\,\mathrm{d}\tau. \tag{A.14}$$

From Eqs. (A.13) and (A.14),

$$\begin{aligned}\boldsymbol{\lambda}(t) = {} & [\boldsymbol{\theta}_{21}(t,t_1) + \boldsymbol{\theta}_{22}(t,t_1)\boldsymbol{P}_1][\boldsymbol{\theta}_{11}(t,t_1) + \boldsymbol{\theta}_{12}(t,t_1)\boldsymbol{P}_1]^{-1}[\boldsymbol{x}(t) - \int_{t_1}^{t} \boldsymbol{\theta}_{11}(t,\tau)\boldsymbol{D}w(\tau)\,\mathrm{d}\tau] \\ & + \int_{t_1}^{t} \boldsymbol{\theta}_{21}(t,\tau)\boldsymbol{D}w(\tau)\,\mathrm{d}\tau. \end{aligned} \tag{A.15}$$

By letting

$$\boldsymbol{S}(t) = [\boldsymbol{\theta}_{21}(t,t_1) + \boldsymbol{\theta}_{22}(t,t_1)\boldsymbol{P}_1][\boldsymbol{\theta}_{11}(t,t_1) + \boldsymbol{\theta}_{12}(t,t_1)\boldsymbol{P}_1]^{-1}, \tag{A.16}$$

$$\boldsymbol{\lambda}(t) = \boldsymbol{S}(t)\int_{t}^{t_1} \boldsymbol{\theta}_{11}(t,\tau)\boldsymbol{D}w(\tau)\,\mathrm{d}\tau - \int_{t}^{t_1} \boldsymbol{\theta}_{21}(t,\tau)\boldsymbol{D}w(\tau)\,\mathrm{d}\tau. \tag{A.17}$$

$\boldsymbol{\lambda}(t)$ can be written as:

$$\boldsymbol{\lambda}(t) = \boldsymbol{S}(t)\boldsymbol{x}(t) + \boldsymbol{f}(t). \tag{A.18}$$

Next, let us obtain the relations for $\boldsymbol{S}(t)$ and $\boldsymbol{f}(t)$.
Let $\boldsymbol{\theta}_1 = [\boldsymbol{\theta}_{11}(t,t_1) + \boldsymbol{\theta}_{12}(t,t_1)\boldsymbol{P}_1]$, $\boldsymbol{\theta}_2 = [\boldsymbol{\theta}_{21}(t,t_1) + \boldsymbol{\theta}_{22}(t,t_1)\boldsymbol{P}_1]$, then,

$$\begin{aligned} \boldsymbol{S}(t) &= \boldsymbol{\theta}_2\boldsymbol{\theta}_1^{-1}, \\ \boldsymbol{\theta}_1' &= -\boldsymbol{P}\boldsymbol{\theta}_{11}(t,t_1) - \boldsymbol{B}\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{\theta}_{21}(t,t_1) + (\boldsymbol{A}\boldsymbol{\theta}_{12}(t,t_1) - \boldsymbol{B}\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{\theta}_{22}(t,t_1))\boldsymbol{R} \\ &= \boldsymbol{A}(\boldsymbol{\theta}_{11}(t,t_1) + \boldsymbol{\theta}_{12}(t,t_1)\boldsymbol{P}_1) - \boldsymbol{B}\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}(\boldsymbol{\theta}_{21}(t,t_1) + \boldsymbol{\theta}_{22}(t,t_1)\boldsymbol{P}_1) = \boldsymbol{A}\boldsymbol{\theta}_1 - \boldsymbol{B}\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{\theta}_2, \end{aligned}$$

$$\begin{aligned}\boldsymbol{\theta}_2' &= \boldsymbol{\theta}_{21}(t,t_1)' + \boldsymbol{\theta}_{22}(t,t_1)'\boldsymbol{P}_1 = -\boldsymbol{P}\boldsymbol{\theta}_{11}(t,t_1) - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\theta}_{21}(t,t_1) - (\boldsymbol{P}\boldsymbol{\theta}_{12}(t,t_1) - \boldsymbol{A}\boldsymbol{\theta}_{22}(t,t_1))\boldsymbol{P}_1 \\ &= -\boldsymbol{P}(\boldsymbol{\theta}_{11}(t,t_1) + \boldsymbol{\theta}_{12}(t,t_1)\boldsymbol{P}_1) - \boldsymbol{A}^{\mathrm{T}}(\boldsymbol{\theta}_{21}(t,t_1) + \boldsymbol{\theta}_{22}(t,t_1)\boldsymbol{P}_1) = -\boldsymbol{P}\boldsymbol{\theta}_1 - \boldsymbol{A}\boldsymbol{\theta}_2^{\mathrm{T}}. \end{aligned} \tag{A.19}$$

Then,

$$\begin{aligned}\boldsymbol{S}'(t) &= \boldsymbol{\theta}_2'\boldsymbol{\theta}_1^{-1} - \boldsymbol{\theta}_2\boldsymbol{\theta}_1^{-1}\boldsymbol{\theta}_1'\boldsymbol{\theta}_1^{-1} = (-\boldsymbol{P}\boldsymbol{\theta}_1 - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\theta}_2)\boldsymbol{\theta}_1^{-1} - \boldsymbol{\theta}_2\boldsymbol{\theta}_1^{-1}(\boldsymbol{A}\boldsymbol{\theta}_1 - \boldsymbol{B}\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{\theta}_2)\boldsymbol{\theta}_1^{-1} \\ &= -\boldsymbol{P} - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\theta}_2\boldsymbol{\theta}_1^{-1} - \boldsymbol{\theta}_2\boldsymbol{\theta}_1^{-1}\boldsymbol{A} + \boldsymbol{\theta}_2\boldsymbol{\theta}_1^{-1}\boldsymbol{B}\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{\theta}_2\boldsymbol{\theta}_1^{-1} \\ &= -\boldsymbol{P} - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{S}(t) - \boldsymbol{S}(t)\boldsymbol{A} + \boldsymbol{S}(t)\boldsymbol{B}\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{S}(t). \end{aligned} \tag{A.20}$$

$$\begin{aligned}\boldsymbol{f}'(t) &= -\boldsymbol{S}'(t)\int_{t_1}^{t}\boldsymbol{\theta}_{11}(t,\tau)\boldsymbol{D}w(\tau)\,\mathrm{d}\tau - \boldsymbol{S}(t)\left\{\boldsymbol{\theta}_{11}(t,t)\boldsymbol{D}w(t) + \int_{t_1}^{t}\boldsymbol{\theta}_{11}'(t,\tau)\boldsymbol{D}w(\tau)\,\mathrm{d}\tau\right\} \\ &\quad + \left\{\boldsymbol{\theta}_{21}(t,t)\boldsymbol{D}w(t) + \int_{t_1}^{t}\boldsymbol{\theta}_{21}'(t,\tau)\boldsymbol{D}w(\tau)\,\mathrm{d}\tau\right\} \\ &= \{\boldsymbol{P} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{S}(t) + \boldsymbol{S}(t)\boldsymbol{A} - \boldsymbol{S}(t)\boldsymbol{B}\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{S}(t)\}\left(\int_{t_1}^{t}\boldsymbol{\theta}_{11}(t,\tau)\boldsymbol{D}w(\tau)\,\mathrm{d}\tau\right) - \boldsymbol{S}(t)\boldsymbol{D}w(t) \\ &\quad - \boldsymbol{S}(t)\left\{\int_{t_1}^{t}(\boldsymbol{A}\boldsymbol{\theta}_{11}(t,\tau) - \boldsymbol{B}\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{\theta}_{21}(t,\tau))\boldsymbol{D}w(\tau)\,\mathrm{d}\tau\right\} + \boldsymbol{0} + \int_{t_1}^{t}(-\boldsymbol{P}\boldsymbol{\theta}_{11}(t,\tau) \\ &\quad - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\theta}_{21}(t,\tau))\boldsymbol{D}w(\tau)\,\mathrm{d}\tau\} \\ &= \{\boldsymbol{A}^{\mathrm{T}} - \boldsymbol{S}(t)\boldsymbol{B}\boldsymbol{Q}^{-\mathbf{1}}\boldsymbol{B}^{\mathrm{T}}\}\left\{\boldsymbol{S}(t)\int_{t_1}^{t}\boldsymbol{\theta}_{11}(t,\tau)\boldsymbol{D}w(\tau)\,\mathrm{d}\tau - \int_{t_1}^{t}\boldsymbol{\theta}_{21}(t,\tau)\boldsymbol{D}w(\tau)\,\mathrm{d}\tau\right\} - \boldsymbol{S}(t)\boldsymbol{D}w(t) \\ &= -\{\boldsymbol{A}^{\mathrm{T}} - \boldsymbol{S}(t)\boldsymbol{B}\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}\}\boldsymbol{f}(t) - \boldsymbol{S}(t)\boldsymbol{D}w(t). \end{aligned} \tag{A.21}$$

And, the end conditions are:

$$\begin{aligned}\boldsymbol{S}(t_1) &= [\boldsymbol{\theta}_{21}(t_1,t_1) + \boldsymbol{\theta}_{22}(t_1,t_1)\boldsymbol{P}_1][\boldsymbol{\theta}_{11}(t_1,t_1) + \boldsymbol{\theta}_{12}(t_1,t_1)\boldsymbol{P}_1]^{-1} = [\boldsymbol{0} + \boldsymbol{I}\boldsymbol{P}_1][\boldsymbol{I} + \boldsymbol{0}\boldsymbol{P}_1]^{-1} = \boldsymbol{P}_1, \boldsymbol{f}(t_1) \\ &= \boldsymbol{S}(t_1)\int_{t_1}^{t_1}\boldsymbol{\theta}_{11}(t,\tau)\boldsymbol{D}w(\tau)\,\mathrm{d}\tau - \int_{t_1}^{t_1}\boldsymbol{\theta}_{21}(t,\tau)\boldsymbol{D}w(\tau)\,\mathrm{d}\tau = 0. \end{aligned} \tag{A.22}$$

Therefore,

$$\boldsymbol{u}(t) = -\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}(\boldsymbol{S}(t)\boldsymbol{x}(t) + \boldsymbol{f}(t)), \tag{A.23}$$

where $\boldsymbol{S}(t)$ and $\boldsymbol{f}(t)$, and their end conditions are given by Eqs. (A.20)–(A.22), respectively.

Appendix B. Induction for least input energy

The followings are the induction for the LIE expressed by Eq. (4.28).

The Hamiltonian is defined by:

$$H = \boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{D}w(t) + \boldsymbol{\lambda}(t)^{\mathrm{T}}\{\boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{D}w(t)\}. \tag{B.1}$$

Then, the optimal values satisfy:

$$\boldsymbol{\lambda}'(t) = -\partial H/\partial \boldsymbol{x} = -\boldsymbol{P}\boldsymbol{D}w(t) - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\lambda}(t), \tag{B.2}$$

$$\boldsymbol{0} = -\partial H/\partial \boldsymbol{u} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{\lambda}(t) + 2\boldsymbol{Q}\boldsymbol{u}(t). \tag{B.3}$$

Then,

$$\boldsymbol{u}(t) = -\tfrac{1}{2}\boldsymbol{Q}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{\lambda}(t) \tag{B.4}$$

The end condition for Eq. (B.2) are:

$$\boldsymbol{\lambda}(t_1) = 0.$$

Thus, $\boldsymbol{\lambda}(t)$ is expressed by:

$$\boldsymbol{\lambda}(t) = \int_t^{t_1} \Xi(t-\tau)^{\mathrm{T}}\boldsymbol{PD}w(\tau)\,\mathrm{d}\tau, \tag{B.5}$$

where $\Xi(t) = \exp(t\boldsymbol{A})$.

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